-CHAPTER TWO-

GENERAL DIFFERENTIAL EQUATION FOR HEAT CONDUCTION

1.1 **General differential equation for heat conduction in Cartesian coordinates.**

1.2 **General differential equation for heat conduction in cylindrical coordinates.**

1.3 **General differential equation for heat conduction in spherical coordinates.**

GENERAL DIFFERENTIAL EQUATION FOR HEAT CONDUCTION

1.1 General differential equation for heat conduction in Cartesian coordinates

This is also known as heat diffusion equation or, simply heat equation. Consider a homogeneous body within which there is no bulk motion and heat transfer occurs in this body by conduction. Temperature distribution within the body at any given instant is given by: T(x, y, z, τ). The coordinate system used in this derivation is given in Fig. (1)

Consider a differential volume element dx.dy.dz from within the body as shown. It has six surfaces Further, the body is assumed to be rigid, i.e. negligible work is done on the body by external mechanical forces.

Let us make an energy balance on this differential element. Let us list out the various energy terms involved: first, there is energy conducted into the element; second, there is energy conducted out of the element; third, for generality, let there be energy generated within the element, say, due to chemical reaction or nuclear fission, etc. Net heat conducted into the element in conjunction with the heat generated within the element, will obviously cause an increase in the energy content (or the internal energy) of the element.
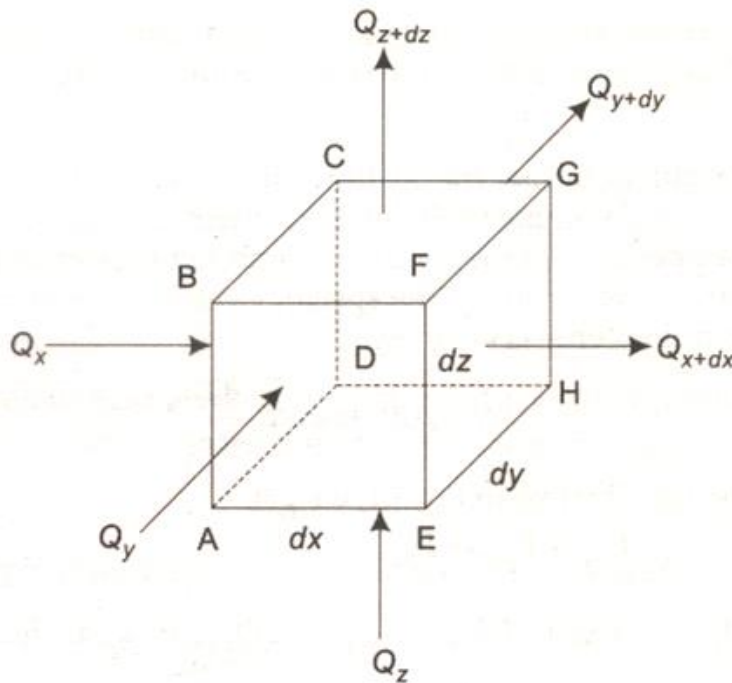

Figure (1)

We can write it mathematically as

$$E_{in} - E_{out} + E_{gen} = E_{st} \ldots \ldots \ldots .1$$

where,

E_{in}= energy entering the control volume per unit time.

E_{out} = energy leaving the control volume per unit time.

E_{gen} = energy generated within the control volume per unit time

E_{st} = energy storage within the control volume per unit time.

To calculate E_{in}. Energy enters the differential control volume from all the three sides by conduction only, since the control volume is embedded within the body considered.

Let the energy entering the control volume in the X-direction through face ABCD be Qx. Similarly, Qy and Qz enter the control volume from the faces ABFE and DAEH as shown in the Fig. (1).

$$E_{in} = Qx + Qy + Qz \dots\dots\dots\dots\dots .2$$

To calculate E_{out}. Energy entering the control volume in the X-direction at face ABCD leaves the control volume at the opposite face EFGH. This is designated as Q_{x+dx} Similarly, Q_{y+dy} and Q_{z+dz} leave the control volume from the surfaces opposite to the ones at which they entered. Therefore, we write,

$$E_{out} = Q_{x+dx} + Q_{y+dy} + Q_{z+dz} \dots\dots\dots\dots .3$$

Now, from calculus, we know that Q_{x+dx} etc. can be expressed by a Taylor series expansion, where, neglecting the higher order terms, we can write,

$$Q_{x+dx} = Q_x + \frac{\partial Q_x}{\partial x} . dx \dots\dots\dots\dots\dots .4.1$$

$$Q_{y+dy} = Q_y + \frac{\partial Q_y}{\partial y} . dy \dots\dots\dots\dots\dots .4.2$$

$$Q_{z+dz} = Q_z + \frac{\partial Q_z}{\partial z} . dz \dots\dots\dots\dots\dots .4.3$$

To calculate E_{gen}. Let there be uniform heat generation within the volume at a rate of q_g (W/m3). Heat generation is a volume phenomenon, i.e. heat is generated throughout the bulk of the body so, note its units (W/m^3). As mentioned earlier, heat may be generated within the body due to passage of an electric current, a chemical reaction, nuclear fission, etc. Then, for the differential control volume $dx. dy. dz$, we can write,

$$E_{gen} = q_g dx. dy. dz \dots\dots\dots\dots .5$$

to calculate E_{st}. As a result of the net energy flow into the control volume from all the three directions and the heat generated within the control volume itself, internal energy of the control volume increases. This will manifest itself as an increase in the temperature of the control volume. Let the temperature of the control volume increase by dT in time $d\tau$. Then, if ρ is the density and c_p, the specific heat of the material of the control volume, rate of increase of internal energy of control volume is given by:

$$E_{st} = \rho. dx \, dy \, dz. c_p \frac{\partial T}{\partial \tau} \dots\dots\dots\dots .6$$

Now, substituting for all terms in Eq. 1, we get,

$$E_{in} - E_{out} + E_{gen} = E_{st}$$

$$\left(Q_x + Q_y + Q_z\right) - \left(Q_{x+dx} + Q_{y+dy} + Q_{z+dz}\right) + q_g dx. dy. dz = \rho. dx \, dy \, dz. c_p \frac{\partial T}{\partial \tau}$$

$$-\left(\frac{\partial Q_x}{\partial x} . dx + \frac{\partial Q_y}{\partial y} . dy + \frac{\partial Q_z}{\partial z} . dz\right) + q_g dx. dy. dz = \rho \, c_p \frac{\partial T}{\partial \tau} dx \, dy \, dz \dots\dots .7$$

Now let us bring in Fourier's law of heat conduction.

$$Q_x = -kA_x \frac{\partial T}{\partial x} = -kdydz \frac{\partial T}{\partial x} \ldots \ldots \ldots 8.1$$

$$Q_y = -kA_y \frac{\partial T}{\partial y} = -kdxdz \frac{\partial T}{\partial y} \ldots \ldots \ldots 8.2$$

$$Q_z = -kA_z \frac{\partial T}{\partial z} = -kdxdy \frac{\partial T}{\partial z} \ldots \ldots \ldots 8.3$$

Substituting Eq.(8), in Eq.(7), and dividing by $dx.dy.dz$, we obtain,

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + q_g = \rho \, c_p \frac{\partial T}{\partial \tau}$$

$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + q_g = \rho \, c_p \frac{\partial T}{\partial \tau}$$

This is the general form of heat diffusion equation in Cartesian coordinates for time dependent (i.e. unsteady state) heat conduction, with uniform heat generation within the body. This is a very important basic equation for conduction analysis. It has to be solved with appropriate initial and boundary conditions to get the temperature distribution within the body as a function of spatial and time coordinates. Of course, the heat transfer rate is calculated applying the Fourier's law, once the temperature distribution is known.

$$\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{q_g}{k} = \frac{\rho \, c_p}{k}\frac{\partial T}{\partial \tau} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$

$$\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau} \ldots \ldots \ldots 9$$

Where, $\alpha = k/\rho c_p$ is thermal diffusivity.

Solution of general form of heat diffusion equation as given in Eq.9 is rather complicated. However, in many practical applications, we make simplifying assumptions and the resulting equations are easily solved. For example:

1) **Steady state**: This means that the temperature at any position does not change with time, i.e. $\frac{\partial T}{\partial \tau} = 0$

   Eq.9, becomes:

$$\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{q_g}{k} = 0$$

   This is known as Poisson equation and is for steady state, three-dimensional heat conduction with heat generation, with constant thermal conductivity, in Cartesian coordinates.

2) **With no Internal heat generation**: This means that q_g term is zero. So, Eq. 9 becomes,

$$\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$

This is known as Diffusion equation, and it represents time dependent, three-dimensional heat conduction, with no internal heat generation, and with constant thermal conductivity, in Cartesian coordinates.

3) **Steady state, with no Internal heat generation**: This means that q_g and $\frac{\partial T}{\partial \tau}$ are zero. So, Eq. 9 becomes.

$$\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) = 0$$

This is known as Laplace equation, and it represents steady state, three-dimensional heat conduction with no internal heat generation, with constant thermal conductivity, in Cartesian coordinates.

4) **One-dimensional, steady state, with no internal heat generation**: This means that.

$$\left(\frac{\partial^2 T}{\partial y^2} = \frac{\partial^2 T}{\partial z^2} = 0\right), \quad (q_g = 0) and \left(\frac{\partial T}{\partial \tau} = 0\right)$$

So Eq. 9 becomes,

$$\frac{\partial^2 T}{\partial x^2}$$

1.2 General differential equation for heat conduction in cylindrical coordinates

Eq. 9 derived earlier is suitable to analyse heat transfer in regular bodies of rectangular, square or parallelepiped shapes. But, if we have to analyse heat transfer in cylindrical-shaped bodies (which are commonly used in practice), then, working with cylindrical coordinates is more convenient, since in that case, the coordinate axes match with the system boundaries. Nomenclature for cylindrical coordinate system is shown in Fig. 2.
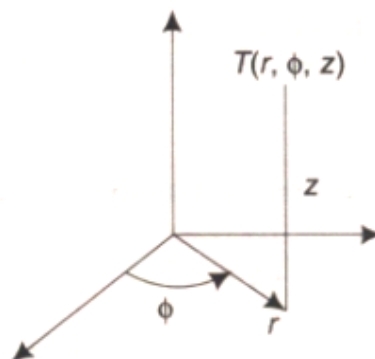

Figure2

Differential equation for heat conduction in cylindrical coordinates may be derived by considering an elemental cylindrical control volume of thickness dr and making an energy balance over this control volume, as was done in the case of Cartesian coordinates, or, coordinates transformation can be adopted; for this purpose, transformation equations are,

$$x = r \cos\emptyset$$
$$y = r \sin\emptyset$$
$$z = z$$
$$\emptyset = tan^{-1}(y/x)$$

The resulting general differential equation in cylindrical coordinates is,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \emptyset^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau} \dots \dots \dots \dots 10$$

Eq. 10 is the general differential equation in cylindrical coordinates, for time dependent, three-dimensional conduction, with constant thermal conductivity and with internal heat generation.

For one-dimensional conduction in r direction only, we get from Eq. 10,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$

$$\frac{1}{r}\left(\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial r^2} + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau} \dots \dots \dots \dots 11$$

And for steady state we get:

$$\frac{1}{r}\left(\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial r^2} + \frac{q_g}{k} = 0 \dots \dots \dots .12$$

1.3 General differential equation for heat conduction in spherical coordinates

To analyse heat transfer in spherical systems, working with spherical coordinates is more convenient, since the coordinate axes match with system boundaries. Nomenclature for the spherical coordinates is shown in Fig. 3.
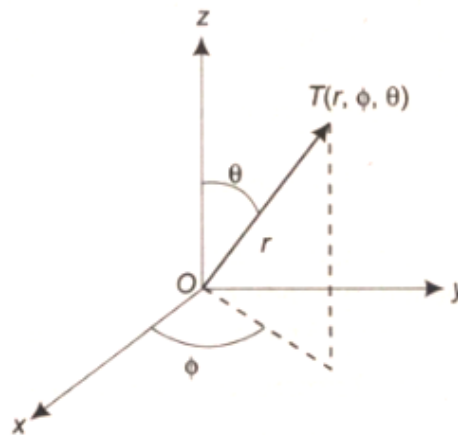

Figure (3)

Differential equation for heat conduction in spherical coordinates may be derived by considering an elemental spherical control volume and making an energy balance over this control volume, as was done in the case of Cartesian and cylindrical coordinates, or, coordinate transformation can be adopted using the following transformation equations,

$$x = r\ sin\theta\ cos\emptyset$$
$$y = r\ sin\theta\ sin\emptyset$$
$$z = r\ cos\theta$$

The resulting general differential equation in spherical coordinates is,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial \theta}\left(sin\theta\ \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 sin\theta}\frac{\partial^2 T}{\partial \emptyset^2} + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau} \dots\dots.13$$

Eq. 13 is the general differential equation in spherical coordinates, for time dependent, three-dimensional conduction, with constant thermal conductivity and with internal heat generation.

For one-dimensional conduction in r direction only, we get from Eq. 13,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau} \dots\dots\dots\dots 14$$

Eq. 14 represents one-dimensional, time dependent conduction in r direction only, with constant k and uniform internal heat generation, in spherical coordinates.

And, for steady state, one-dimensional heat conduction in r direction only, with constant k and uniform heat generation Eq. 14 reduces to,

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{q_g}{k} = 0$$

-CHAPTER THREE-

ONE DIMENSIONAL STEADY STATE HEAT CONDUCTION

2.1 One-Dimensional Heat Conduction, No Heat Generation for Plane Slab

2.2 One-Dimensional Heat Conduction, No Heat Generation for Cylindrical Systems

2.3 One-Dimensional Heat Conduction, No Heat Generation for Spherical Systems

2.4 Heat Flow Through Composite Slabs

2.4.1 Thermal resistances in series

2.4.2 Thermal resistances in parallel

2.4.3 Thermal resistance for cylindrical shell

2.4.4 Thermal resistance for spherical system

2.5.1 Overall Heat Transfer Coefficient

2.5.2 Overall Heat Transfer Coefficient for the Cylindrical System

2.5.3 Overall Heat Transfer Coefficient for the spherical System

2.6 Critical Thickness Of Insulation

2.7.1 Plane Slab With Heat Sources

2.7.2 Cylinder With Heat Sources

ONE-DIMENSIONAL, STEADY STATE HEAT CONDUCTION, WITH NO INTERNAL HEAT GENERATION

2.1 Plane Slab

The governing equation for a plane slab with One-dimensional, steady state heat conduction, with no internal heat generation is:

$$\frac{\partial^2 T}{\partial x^2} = 0 \dots\dots\dots\dots\dots\dots\dots 1$$

Integrating Eq.1 once:

$$\frac{\partial T}{\partial x} = C_1$$

Integrating again:

$$T(x) = C_1 x + C_2 \dots\dots\dots\dots\dots .2$$

Equation 2 is the general solution for the temperature distribution. Values of the two integration constants C_1 and C_2 are obtained from the two boundary conditions namely
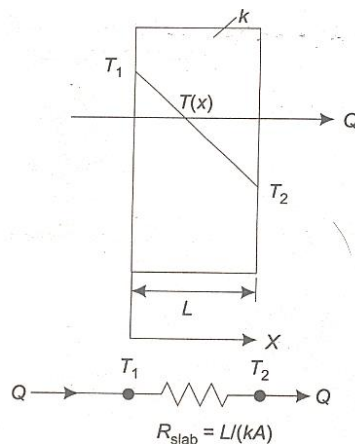

Figure (1)

B.C.(i): $T = T_1$ at $x = 0$
B.C.(ii): $T = T_2$ at $x = L$
From B.C.(i) and Eq.2

$$T(0) = T_1 = C_2$$

From B.C.(ii) and Eq.2

$$T(L) = T_2 = C_1 L + C_2$$

$$T_2 = C_1 L + T_1$$

So:

$$C_1 = (T_2 - T_1)/L$$

Sub. values of C_1 and C_2 1n Eq.2 we get,

$$T(x) = \frac{(T_2 - T_1)}{L} x + T_1 \dots\dots\dots\dots\dots\dots 3$$

Eq. 3 can be written in non dimensional form:

$$\frac{T(x) - T_1}{(T_2 - T_1)} = \frac{x}{L} \ldots\ldots\ldots\ldots\ldots\ldots.4$$

To find the heat flux, apply Fourier law,

$$q = -k\frac{dT}{dx}$$

$$\frac{dT}{dx} = C_1 = (T_2 - T_1)/L$$

So:

$$q = -k\frac{(T_2 - T_1)}{L} = k\frac{(T_1 - T_2)}{L} \quad W/m^2 \ldots\ldots\ldots\ldots.5$$

The heat flow rate is

$$Q = qA = -kA\frac{(T_2 - T_1)}{L} \quad W \ldots\ldots\ldots\ldots.6$$

2.2 Cylindrical Systems

The governing equation for the Cylindrical systems, One-dimensional, steady state heat conduction, with no internal heat generation is:

$$\frac{1}{r}\left(\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial r^2} = 0$$

Multiplying by r, we get.

$$\frac{\partial T}{\partial r} + r\frac{\partial^2 T}{\partial r^2} = 0$$

i.e.

$$\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = 0$$

Integrating,

$$r\frac{\partial T}{\partial r} = C_1$$

$$\frac{\partial T}{\partial r} = \frac{C_1}{r}$$

Integrating again,

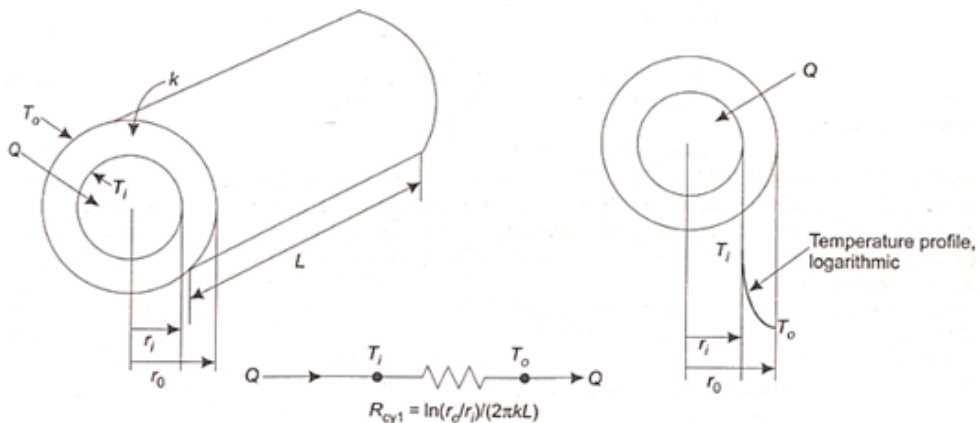$$T(r) = C_1 \ln(r) + C_2 \ldots\ldots\ldots. a$$

Figure (1b)

The equation gives the temperature distribution as a function of radius.

Where C_1 and C_2 are constants of integration, and they are found out by applying the two B.C:

1- $at \quad r = r_i \quad T = T_i$
2- $at \quad r = r_o \quad T = T_0$

B.C. (1) gives, $\qquad\qquad\qquad T(i) = C_1 \ln(r_i) + C_2 \dots\dots b$

B.C. (2) gives, $\qquad\qquad\qquad T(o) = C_1 \ln(r_o) + C_2 \dots\dots c$

Subtracting Eq. c from Eq. b:

$$T_i - T_o = C_1 \ln(r_i/r_o)$$

$$C_1 = \frac{T_i - T_o}{\ln(r_i/r_o)} = \frac{T_o - T_i}{\ln(r_o/r_i)}$$

$$C_2 = T_i - \frac{T_o - T_i}{\ln(r_o/r_i)} \ln(r_i)$$

Substituting C1 and C2 in equation a, we get

$$T(r) = \frac{T_o - T_i}{\ln(r_o/r_i)} \ln(r) + T_i - \frac{T_o - T_i}{\ln(r_o/r_i)} \ln(r_i)$$

$$T(r) = T_i + \frac{T_o - T_i}{\ln(r_o/r_i)} \ln\left(\frac{r}{r_i}\right)$$

$$\frac{T(r) - T_i}{T_o - T_i} = \frac{\ln\left(\frac{r}{r_i}\right)}{\ln\left(\frac{r_o}{r_i}\right)}$$

To find the heat transfer rate:

$$\left[Q = -kA_r \frac{dT}{dr}\right]_{r=r_i} = -k2\pi r_i L \frac{C_1}{r_i}$$

$$Q = -k2\pi r_i L \frac{T_o - T_i}{r_i \ln\left(\frac{r_o}{r_i}\right)}$$

$$Q = k2\pi L \frac{T_i - T_o}{\ln\left(\frac{r_o}{r_i}\right)} \dots\dots d$$

2.3 Spherical Systems

Spherical system is one of the most commonly used geometries in industry. It finds its applications as storage tanks, reactors, etc. in petrochemical, refineries and cryogenic industries. Sphere has minimum surface area for a given volume and material requirement to manufacture a sphere is minimum compared to other geometries.

The general differential equation in spherical coordinates, for steady state, one-dimensional heat conduction in r direction only, with constant k, no heat generation and uniform heat generation is:

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} = 0$$

The non-dimensional form for the temperature distribution equation is:

$$\frac{T(r) - T_i}{T_o - T_i} = \frac{\frac{1}{r} - \frac{1}{r_i}}{\frac{1}{r_o} - \frac{1}{r_i}} = \frac{r_o}{r} * \left(\frac{r - r_i}{r_o - r_i}\right)$$

The heat transfer rate equation is:

$$Q = \frac{\Delta T}{R_{sph}} = \frac{T_i - T_o}{\frac{r_o - r_i}{4\pi k r_o r_i}}$$

2.4 Heat Flow Through Composite Slabs

Heat transfer through a composite slab, consisting of 2 or 3 layers of materials of different thermal conductivities. This is a very common application, e.g. in the case of insulation of furnace walls, insulation of walls of buildings, refrigerators, cold storage plants, hot water tanks, etc.

While solving heat transfer problems in composite slabs under steady state conditions, it is convenient to use the thermal resistance concept.

Consider a composite slab consisting of three layers 1,2 and 3 as shown in Fig.2. Let the thicknesses of the three layers be L_1, L_2 and L_3, respectively; also, the respective thermal conductivities are k_1, k_2 and k_3.

Fluid at a temperature T_a flows on the surface with a convective heat transfer coefficient of h_a and, a fluid at a temperature of T_b flows with a convective heat transfer coefficient of h_b, as shown. Let T_a be higher than T_b, so that steady state heat transfer rate Q is from left to right as indicated in the Fig. 2.

Assumptions:
1. Steady state, one-dimensional heat conduction.
2. No internal heat generation.
3. Constant thermal conductivities k_1, k_2 and k_3.
4. There is perfect thermal contact between layers, i.e. there is no temperature drop at the interface and the temperature profile is continuous.

Since it is a case of steady state conduction with no internal heat generation, it is clear from the First law that heat flow rate Q, through each layer is the same. Referring to Fig.2, it may be seen that heat flows from the fluid at temperature T_a to the left surface of slab 1 by convection, then by conduction through slabs 1, 2 and 3, and then, by convection from the right surface of slab 3 to the fluid at temperature T_b.

Let the area of the slab normal to the heat flow direction be $A(m^2)$. Now, considering each case by turn.

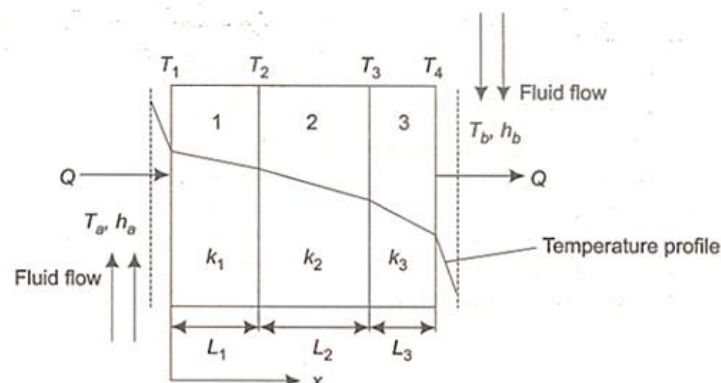

Figure (2)

Convection on the left surface of slab1

$$Q = h_a A(T_a - T_1)$$

So

$$T_a - T_1 = \frac{Q}{h_a A} \ldots \ldots \ldots \ldots . a$$

Conduction through slab1

$$Q = \frac{k_1 A(T_1 - T_2)}{L_1}$$

So

$$(T_1 - T_2) = \frac{QL_1}{k_1 A} \ldots \ldots \ldots \ldots \ldots . b$$

Conduction through slab2

$$Q = \frac{k_2 A(T_2 - T_3)}{L_2}$$

So

$$(T_2 - T_3) = \frac{QL_2}{k_2 A} \ldots \ldots \ldots \ldots \ldots . c$$

Conduction through slab3

$$Q = \frac{k_3 A(T_3 - T_4)}{L_3}$$

So

$$(T_3 - T_4) = \frac{QL_3}{k_3 A} \ldots \ldots \ldots \ldots \ldots . d$$

Convection on the left surface of slab3

$$Q = h_b A(T_4 - T_b)$$

So

$$T_4 - T_b = \frac{Q}{h_b A} \ldots \ldots \ldots \ldots . e$$

Adding Eq. a, b, c, d and e, we get:

$$T_a - T_b = Q \left[\frac{1}{h_a A} + \frac{L_1}{k_1 A} + \frac{L_2}{k_2 A} + \frac{L_3}{k_3 A} + \frac{1}{h_b A} \right] \ldots \ldots \ldots \ldots . f$$

i.e.

$$T_a - T_b = Q[R_a + R_1 + R_2 + R_3 + R_b] \ldots \ldots \ldots \ldots . g$$

where,

R_a = convective resistance at left surface of slab 1,

R_1 = conductive resistance of slab 1,

R_2 = conductive resistance of slab 2,

R_3 = conductive resistance of slab 3, and

R_b = convective resistance at right surface of slab 3.

So, we write Eq. g as:

$$Q = \left[\frac{T_a - T_b}{R_a + R_1 + R_2 + R_3 + R_b} \right] \ldots \ldots \ldots \ldots .7$$

Now, observe the analogy with Ohm's law. Refer to the Fig.3 for the equivalent thermal circuit. It is clear that $(T_a - T_b)$ is the total temperature potential, Q is the heat current flowing and the total resistance is the sum of the individual five resistances which are in series.

2.4.1 Thermal resistances in series, we have,

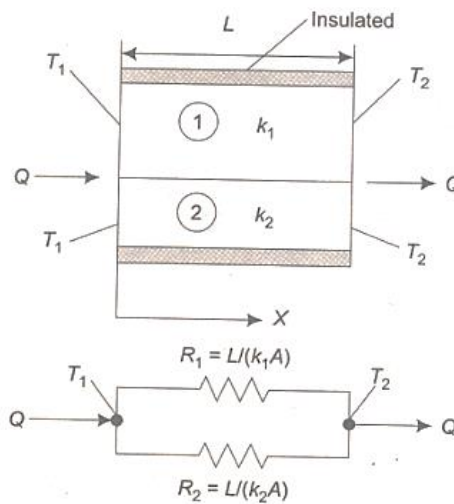$$R_{tot} = \sum R \ldots\ldots\ldots\ldots\ldots.8$$

Figure (3)

2.4.2 Thermal resistances in parallel

Thermal resistances may be arranged in parallel too, as shown in Fig.3.

From the analogy with the electrical circuit, when the resistances are in parallel, the total resistance is given by:

$$\frac{1}{R_{tot}} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{\dfrac{L}{k_1 A}} + \frac{1}{\dfrac{L}{k_2 A}}$$

$$R_{tot} = \frac{R_1 R_2}{R_1 + R_2} \ldots\ldots\ldots\ldots 9$$

Thermal resistances in series and parallel: general case of thermal resistances arranged in series and parallel is shown in fig. 4.
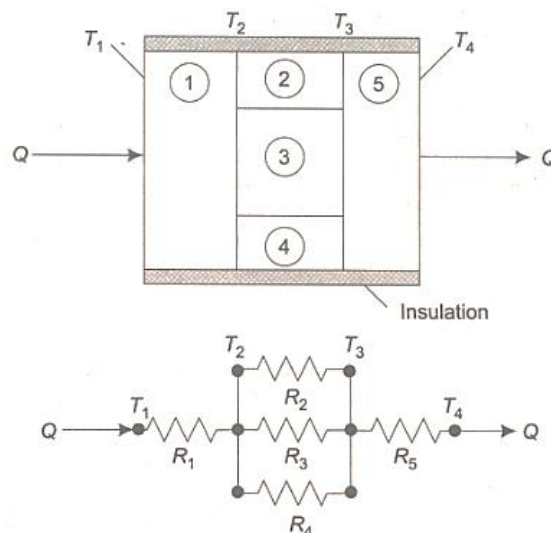

Figure (4)

Applying the rules of electrical circuit for series and parallel, we have,

$$Q = \frac{\Delta T}{R_1 + R_{eff} + R_2} = \frac{T_1 - T_4}{R_1 + R_{eff} + R_2} \dots \dots \dots .10$$

Where R_{eff} is the effective resistance of the three resistances R2, R3 and R4 in parallel as shown in fig. 4.

$$\frac{1}{R_{eff}} = \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} \dots \dots \dots \dots \dots ..11$$

2.4.3 Thermal resistance for conduction for cylindrical shell

From Eq. d

$$Q = k2\pi L \frac{T_i - T_o}{\ln\left(\frac{r_o}{r_i}\right)} \dots \dots .. d$$

Writing Eq. d in a form analogous to ohm's law:

$$Q = \frac{\Delta T}{R_{cyl}} = \frac{T_i - T_o}{\frac{\ln\left(\frac{r_o}{r_i}\right)}{k2\pi L}}$$

we observe that thermal resistance for conduction for cylindrical shell is given by,

$$R_{cyl} = \frac{\ln\left(\frac{r_o}{r_i}\right)}{2\pi k L} \dots \dots \dots \dots \dots 12$$

2.4.4 Thermal resistance for spherical system

$$Q = \frac{\Delta T}{R_{sph}} = \frac{T_i - T_o}{\frac{r_o - r_i}{4\pi k r_o r_i}}$$

$$R_{sph} = \frac{r_o - r_i}{4\pi k r_o r_i} = \frac{1}{4\pi k}\left[\frac{1}{r_i} - \frac{1}{r_o}\right]$$

2.5.1 Overall Heat Transfer Coefficient, U (W/(M²C))

Consider the case of a furnace where heat is transferred by the hot gases to the inside surface by convection, then by conduction through one, two or three layers of brick and insulation, and finally to ambient air by convection at the outermost surface. This situation is represented in Fig. 2.

Now, in most of the practical cases, temperature of the hot gases (T_a) and that of the ambient (T_b) are known; intermediate temperatures are not known. *We would like to have the heat transfer given by a simple relation of the form:*

$$Q = UA(T_a - T_b) = UA\Delta T \dots \dots \dots 13$$

where, Q is the heat transfer rate (W), A is the area of heat transfer perpendicular to the direction of heat transfer, and $(T_a - T_b) = \Delta T$ is the overall temperature difference.

Our problem is to derive a relation for U.

Now, we have from Eq. 7,

$$Q = \left[\frac{T_a - T_b}{R_a + R_1 + R_2 + R_3 + R_b}\right] \dots \dots \dots \dots ..7$$

Comparing Eq. 7 and Eq. 13;

$$Q = UA(T_a - T_b) = \left[\frac{T_a - T_b}{R_a + R_1 + R_2 + R_3 + R_b}\right] = \frac{T_a - T_b}{\sum R_{th}}$$

So

$$UA = \frac{1}{\sum R_{th}} \dots\dots\dots 14$$

$$U = \frac{1}{A \sum R_{th}} \quad , \frac{W}{m^2 C} \dots\dots\dots 15$$

$$U = \frac{1}{\frac{1}{h_a} + \frac{L_1}{k_1} + \frac{L_2}{k_2} + \frac{L_3}{k_3} + \frac{1}{h_b}} \dots\dots.16$$

Remember the expression for U as given by Eq. 15; it is easier and is applicable when we deal with other geometries, too.

Concept of overall heat transfer coefficient is particularly useful in heat exchanger designs. Consider a heat exchanger where a hot fluid flows on one side of a heat exchanger wall and a cold fluid flows on the other side. Then, heat transfer is by convection on the hot side, by conduction across the separating wall and again by convection on the cold side, overall heat transfer coefficient is obtained by applying Eq. 15.

Values of overall heat transfer coefficient for many practical cases are tabulated in handbooks.

2.5.2 Overall Heat Transfer Coefficient for the Cylindrical System

Referring to Fig. 5, it is clear that heat transfer occurs from hot fluid at T_a to the inner cylinder by convection, then through the inner and outer cylindrical shells by conduction and then to the outer cold fluid at T_b by convection. We would rather like to write the heat transfer rate in terms of the known overall temperature difference, as follows,

$$Q = UA(T_a - T_b) = UA\Delta T$$

where U is an overall heat transfer coefficient and A is the area normal to the direction of heat flow. In the case of a plane slab, A was a constant with x; however, in the case of a cylindrical system, area normal to the direction of heat flow is $2\pi r L$, and clearly, this varies with r. Therefore, while dealing with cylindrical systems, we have to specify as to which area U is based on, i.e. whether it is based on inside area or outside area. (Generally, U is based on outside area since pipes are specified on outside diameters.) We write.

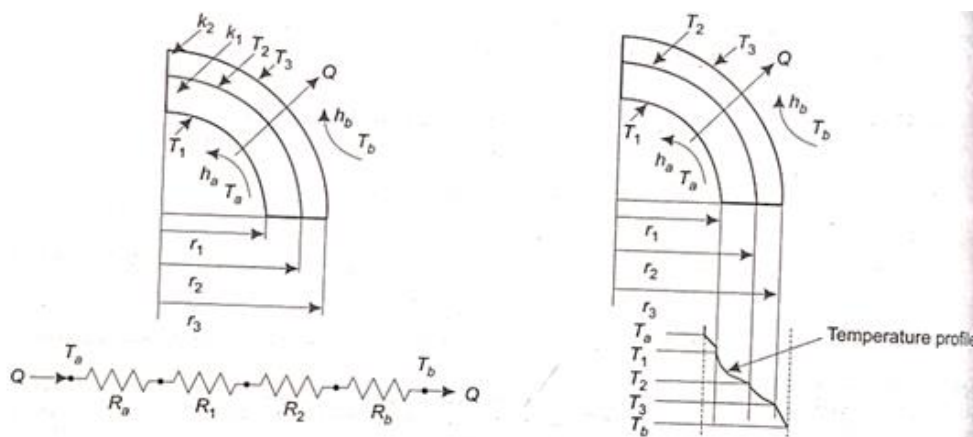$$Q = U_i A_i (T_a - T_b) = U_o A_o (T_a - T_b)$$

Figure (5)

where,

U_i = overall heat transfer coefficient based on inside area

U_o = overall heat transfer coefficient based on outside area

A_i = heat transfer area on inside

A_o = heat transfer area on outside

Comparing with Eq. 7, we get,

$$Q = \frac{T_a - T_b}{\sum R_{th}} = U_i A_i (T_a - T_b) = U_o A_o (T_a - T_b)$$

So

$$U_i A_i = U_o A_o = \frac{1}{\sum R_{th}} \ldots\ldots\ldots.17$$

Therefore

$$U_i = \frac{1}{A_i \sum R_{th}}$$

$$U_o = \frac{1}{A_o \sum R_{th}}$$

$$U_o = \frac{1}{A_o \sum R_{th}} = \frac{1}{2\pi r_3 L * \left[\frac{1}{2\pi r_1 L h_a} + \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi k_1 L} + \frac{\ln\left(\frac{r_3}{r_2}\right)}{2\pi k_2 L} + \frac{1}{2\pi r_3 L h_b} \right]}$$

$$U_o = \frac{1}{\left[\left(\frac{r_3}{r_1}\right)\frac{1}{h_a} + \left(\frac{r_3}{k_1}\right)\ln\left(\frac{r_2}{r_1}\right) + \left(\frac{r_3}{k_2}\right)\ln\left(\frac{r_3}{r_2}\right) + \frac{1}{h_b} \right]}$$

Once the total thermal resistance $\sum R$ is calculated, U_i, or U_o is easily found out from Eq.17. The concept of overall heat transfer coefficient in cylindrical systems is often useful in heat exchanger designs, since cylindrical geometry is a popular choice in heat exchangers.

2.5.3 Overall Heat Transfer Coefficient for the spherical System

$$Q = U_i A_i (T_a - T_b) = U_o A_o (T_a - T_b)$$

where,

U_i = overall heat transfer coefficient based on inside area

U_o = overall heat transfer coefficient based on outside area

A_i = heat transfer area on inside

A_o = heat transfer area on outside

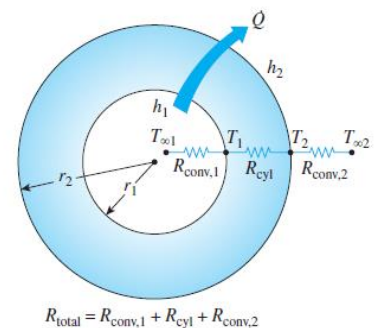$$Q = \frac{T_a - T_b}{\sum R_{th}} = U_i A_i (T_a - T_b) = U_o A_o (T_a - T_b)$$

So

$$U_i A_i = U_o A_o = \frac{1}{\sum R_{th}}$$

Therefore

$$U_i = \frac{1}{A_i \sum R_{th}}$$

$$U_o = \frac{1}{A_o \sum R_{th}}$$

$$U_o = \frac{1}{A_o \sum R_{th}} = \frac{1}{4\pi r_1^2 * \left[\dfrac{1}{4\pi h_a r_1^2} + \dfrac{\ln\left(\frac{r_2}{r_1}\right)}{4\pi h_b r_3^2} + \dfrac{\ln\left(\frac{r_3}{r_2}\right)}{2\pi k_2 L} + \dfrac{1}{2\pi r_3 L h_b}\right]}$$

$$U_o = \frac{1}{\left[\dfrac{1}{h_a}\left(\frac{r_3}{r_1}\right)^2 + \dfrac{1}{h_b} + \dfrac{r_3^2(r_2 - r_1)}{k_1 r_1 r_2} + \dfrac{r_3(r_3 - r_2)}{k_2 r_2}\right]}$$

Note: the above calculations give U_i and U_o in term of inside and outside radii. You need not to memorize them. To calculate U_i or Uo while solving numerical problem just remember Eq. 17

2.6 Critical Thickness Of Insulation

Let us consider a layer of insulation which might be installed around a circular pipe, as shown in Figure 7. The inner temperature of the insulation is fixed at Ti, and the outer surface is exposed to a convection environment at $T\infty$. From the thermal network the heat transfer is

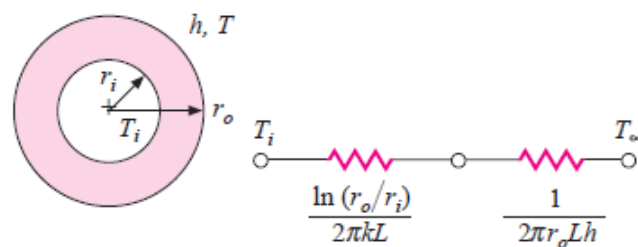$$q = \frac{2\pi L(T_i - T_\infty)}{\dfrac{\ln(r_o/r_i)}{k} + \dfrac{1}{r_o h}}$$

Figure (7)

Now let us manipulate this expression to determine the outer radius of insulation r_o, which will maximize the heat transfer. The maximization condition is

$$\frac{dq}{dr^o} = 0 = \frac{-2\pi L(T_i - T_\infty)\left(\dfrac{1}{r_o k} - \dfrac{1}{r_o^2 h}\right)}{\left[\dfrac{\ln(r_o/r_i)}{k} + \dfrac{1}{r_o h}\right]^2}$$

which gives the result

$$r_o = \frac{k}{h}$$

The last equation expresses the critical-radius-of-insulation concept. If the outer radius is less than the value given by this equation, then the heat transfer will be **increased** by adding more insulation. For outer radii greater than the critical value an increase in insulation thickness will cause a **decrease** in heat transfer. The central concept is that for sufficiently small values of h the convection heat loss may actually increase with the addition of insulation because of increased surface area.

ONE-DIMENSIONAL, STEADY STATE HEAT CONDUCTION, WITH INTERNAL HEAT GENERATION

A number of interesting applications of the principles of heat transfer are concerned with systems in which heat may be generated internally. Nuclear reactors are one example; electrical conductors and chemically reacting systems are others.

2.7.1 Plane Slab With Heat Sources

Consider a plane slab of thickness 2L as shown in Fig. 8. Other dimensions of the slab are comparatively large, so that heat transfer may be considered as one-dimensional in the x-direction, as shown.

The slab has a constant thermal conductivity k, and a uniform internal heat generation rate of q_g (W/m^3). Both the sides of the slab are maintained at the same, uniform temperature of T_w. Then, it is clear that maximum temperature will occur at the centre line, since the heat has to flow from the centre outwards. Therefore it is advantageous to select the origin of the rectangular coordinate system on the centre line, as shown.

Let us analyse this case for temperature distribution within the slab and the heat transfer to the sides.

Assumptions:

1. One-dimensional conduction i.e. thickness L is small compared to the dimensions in the y and z directions.
2. Steady state conduction, i.e. temperature at any point within the slab does not change with time: of course, temperatures at different points within the slab will be different.
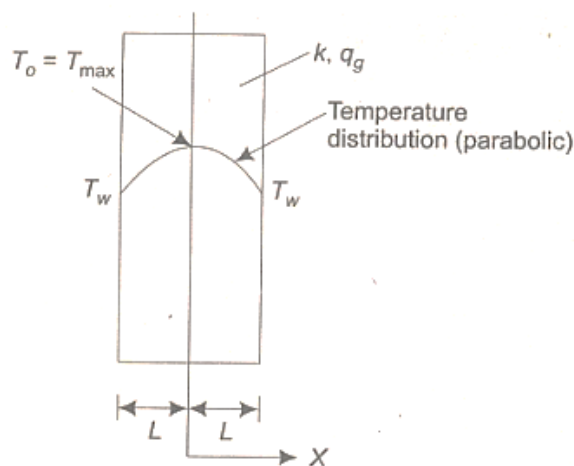3. Uniform internal heat generation rate, q_g (W/m^3)

Figure (8)

4. Material of the slab is homogeneous (i.e. constant density) and isotropic (i.e. value of k is same in all directions).

We wish to find out the temperature field within the slab and then the heat flux at any point

We start with the general differential equation in Cartesian coordinates, for the above-mentioned assumptions.

$$\frac{\partial^2 T}{\partial x^2} + \frac{q_g}{k} = 0 \dots\dots 1.1$$

B.C.:
1) $at\ x = 0, dT/dx = 0$ since the temperature is maximum at the center line.
2) $at\ x = \bar{+}L,\ T = T_w$

Integrating Eq. 1.1

$$\frac{dT}{dx} = \frac{-q_g x}{k} + C_1 \dots\dots\dots\dots a$$

Integrating again,

$$T(x) = \frac{-q_g x^2}{2k} + C_1 x + C_2 \dots\dots\dots 1.2$$

Applying B.C. (1) to Eq. a:

$$C_1 = 0$$

Applying B.C. (2) to Eq. 1.2:

$$T_w = \frac{-q_g L^2}{2k} + C_2$$

$$C_2 = T_w + \frac{q_g L^2}{2k}$$

Sub. C_1 and C_2 in Eq. 1.2:

$$T(x) = \frac{-q_g x^2}{2k} + T_w + \frac{q_g L^2}{2k}$$

$$T(x) = T_w + \frac{q_g}{2k}(L^2 - x^2) \dots\dots\dots 1.3$$

Where, **L is half thickness of the slab** (Remember this)
Also by observation, $T = T_{max}$ $at\ x = 0$.
Then putting $x = 0$ in Eq. 1.3:

$$T_{max} = T_w + \frac{q_g L^2}{2k} \dots\dots\dots\dots 1.4$$

Then from Eq. 1.3 and Eq. 1.4, we get:

$$\frac{T - T_w}{T_{max} - T_w} = \frac{L^2 - x^2}{L^2} = 1 - \left(\frac{x}{L}\right)^2$$

Heat transfer:

In the case of a slab with no internal heat generation, heat flux was the same at every point within the slab, since dT/dx was a constant and independent of x. However, when there is heat generation, dT/dx is not independent of x (see Eq. a), and obviously, heat flux , $q\ (= -kAdT/dx)$ varies from point to point along x within the slab, for the heat transfer at both the surfaces.

$$Q = -kA(dT/dx) \quad at\ x = L$$

By substituting Eq. a, we get:

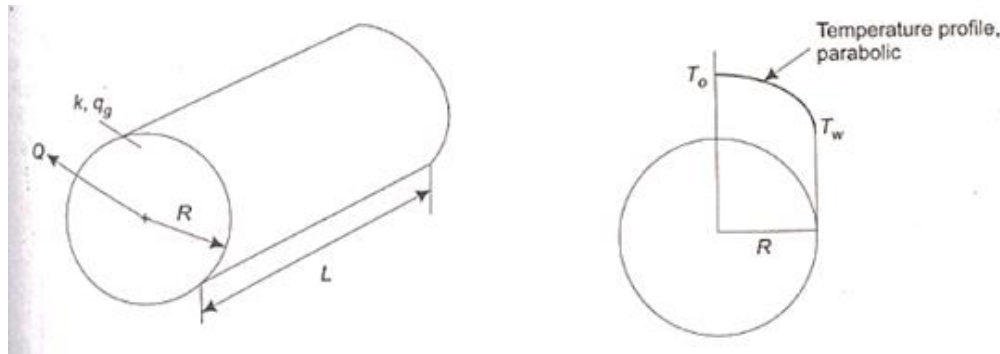$$Q = -kA\left[\frac{-q_g x}{k}\right] \quad at\ x = L$$

$$Q = +q_g AL$$

2.7.2 Cylinder With Heat Sources

consider a solid cylinder of radius, R and length, L. There is uniform heat generation within its volume at a rate of q_g (W/m^3). Let the thermal conductivity, k be constant.
See the Figure.

We would like to analyse this system for temperature distribution and maximum temperature attained.

Assumption:

1. Steady state conduction
2. one-dimension conduction, in the r direction only
3. Homogeneous, isotropic material with constant k
4. Uniform internal heat generation rate. q_g (W/m^3).

With the above stipulations, the general differential equation in cylindrical coordinates reduces to:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{q_g}{k} = 0 \dots \dots .. b$$

Multiplying by r.

$$r\frac{\partial^2 T}{\partial r^2} + \frac{\partial T}{\partial r} = \frac{-q_g r}{k}$$

Integrating:

$$r\frac{\partial T}{\partial r} = \frac{-q_g r^2}{2k} + C_1$$

$$\frac{\partial T}{\partial r} = \frac{-q_g r}{2k} + \frac{C_1}{r} \dots \dots \dots .. c$$

Integrating again:

$$T(r) = \frac{-q_g r^2}{4k} + C_1 \ln(r) + C_2 \dots \dots \dots \dots \dots ..1.5$$

C1 and C2, the constants of integration are obtained by applying the boundary conditions
B.C. are:

1) $at\ r = 0, dT/dr = 0$ i.e. at the temperature at the center of the cylinder,   temperature is finite and maximum ($T_o = T_{max}$) because heat flow from inside to outside.
2) $at\ r = R\ i.e.\ at\ the\ surfuce, T = T_w$

From B.C.1 and Eq. c, we get: $C_1 = 0$

From B.C.2 and Eq. 1.5, we get:

$$T_w = \frac{-q_g R^2}{4k} + C_2$$

$$C_2 = T_w + \frac{q_g R^2}{4k}$$

Sub. C_1 and C_2 in Eq. 1.5:

$$T(r) = \frac{-q_g r^2}{4k} + T_w + \frac{q_g R^2}{4k}$$

$$T(r) = T_w + \frac{q_g}{4k}\left(R^2 - r^2\right) \dots\dots 1.6$$

Maximum temperature occurs at the centre, because of symmetry considerations (i.e. heat flows from the center radially outward in all directions; therefore, temperature at the centre must be a maximum.)
Therefore, putting r = 0 in Eq. 1.6:

$$T_{max} = T_w + \frac{q_g R^2}{4k} \dots\dots\dots 1.7$$

From Eq. 1.6 and 1.7,

$$\frac{T - T_w}{T_{max} - T_w} = 1 - \left(\frac{r}{R}\right)^2$$

-CHAPTER FOUR-

HEAT TRANSFER FROM EXTENDED SURFACES (FINS)

4.1   Introduction

4.2   Fins Of Uniform Cross Section-Governing Differential Equation

4.3   Case (1) Infinitely Long Fin

4.4   Case (2) Fin Of Finite Length With Insulated End

4.5   Case (3) Fin Of Finite Length Losing Heat From Its End By Convection

4.6   Performance Of Fins

4.6.1 Fin Efficiency

4.6.2 Fin Effectiveness (ε_f)

HEAT TRANSFER FROM EXTENDED SURFACES (FINS)

4.1 Introduction

Fins are generally used to enhance the heat transfer from a given surface.

Consider a surface losing heat to the surroundings by convection. Then, the heat transfer rate Q, is given by Newton's law of cooling:

$$Q = h\,A\,(T_o - T_a)$$

Where,

h = heat transfer coefficient between the surface and the ambient

A = exposed area of the surface

T_o = temperature of the surface, and

T_a = temperature of the surroundings.

So if we need to increase the heat transfer rate from the surface, we can:

1. increase the temperature potential, (To - Ta); but, this may not be possible always since both these temperature may not be in our control.
2. increase the heat transfer coefficient h; this also may not be always possible or it may need installing an external fan or pump to increase the fluid velocity and this may involve cost consideration, or
3. increase the surface area A; in fact, this is the solution generally adopted. Surface area is increased by adding an 'extended surface' (or, fin) to the 'base surface' by extruding, welding or by simply fixing it mechanically.

Adding of fins can increase the heat transfer from the surface by several folds, e.g. an automobile radiator has thin sheets fixed over the tubes to increase the area several folds and thus increase the rate of heat transfer

Generally, fins are fixed on that side of the surface when the heat transfer coefficient is low. Heat transfer coefficient are lower for gases as compared to liquids. Therefore, one can observe that fins are fixed on the outside of the tubes in a car radiator, where cooling liquid flows inside the tubes and air flows on the outside across fins.

Likewise, in the condenser of a household refrigerator, freon flows inside the tubes and the fins are fixed on the outside of these tubes to enhance the heat transfer rate.

Application areas of fins are:

1. Radiators for automobiles
2. Heat exchangers of a wide variety, used in different industries
3. Cooling of electric motors, transformers, etc.
4. Cooling of electronic equipments, chips, I.C. boards etc.

Types of fins:

There are innumerable types of fins used in practice some of the more common types are shown in Fig.

Fins of rectangular(a), circular, triangular(d), trapezoidal(c) and conical(g) sections are some of the types commonly used.

(a)   (b)   (c)   (d)
(e)   (f)   (i)   (h)   (g)
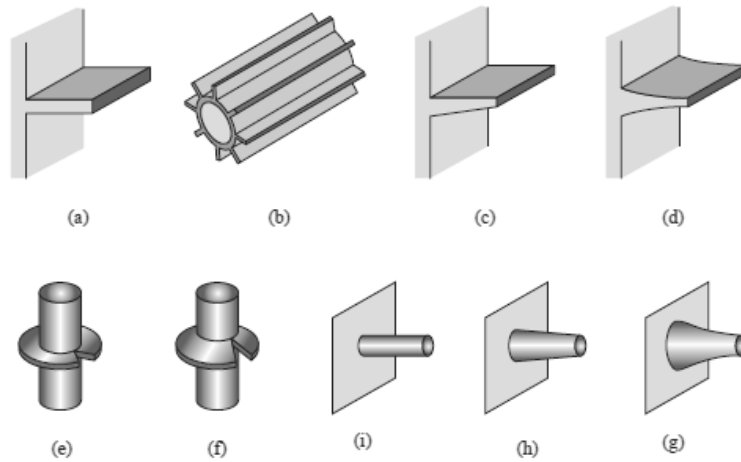
Determination of heat transfer in fins requires information about the temperature profile in the fin. We get the differential equation describing the temperature distribution in the fin by the usual procedure of writing an energy balance for a differential volume of the fin.

4.2 Fins of Uniform Cross Section (Rectangular or Circular)-Governing Differential Equation

Let us analyst' heat transfer in a fin of rectangular cross section. Same analysis will be valid for a fin of circular cross section also.

Consider a fin of rectangular cross section attached to the base surface, as shown in Fig.

Let L be the length of fin,

w, its width and

t its thickness.

Let P be the perimeter = 2 (w + t).

Let A_c be the area of cross section and

T_o the temperature at the base, as shown.

Assumptions:
1. Steady state conduction, with no heat generation in the fin
2. Thickness t is small compared to length L and width w, i.e. one-dimensional conduction in the X-direction only.
3. Thermal conductivity, k of the fin material is constant.
4. Uniform heat transfer coefficient h, over the entire length of fin.
5. No bond resistance in the joint between the fin and the base wall, and
6. Negligible radiation effect.

Base temperature, T_o is higher than the ambient temperature, T_a Temperature will drop along the fin from the base to the tip of the fin, as shown in Fig. Heat transfer will occur by

conduction along the length of the fin and by convection, with a heat transfer coefficient h, from the surface of the fin to the ambient.

Our aim is to derive a differential equation governing the temperature distribution in the fin. Once we get the temperature field, heat flux at any point can easily be obtained by applying Fourier's law.
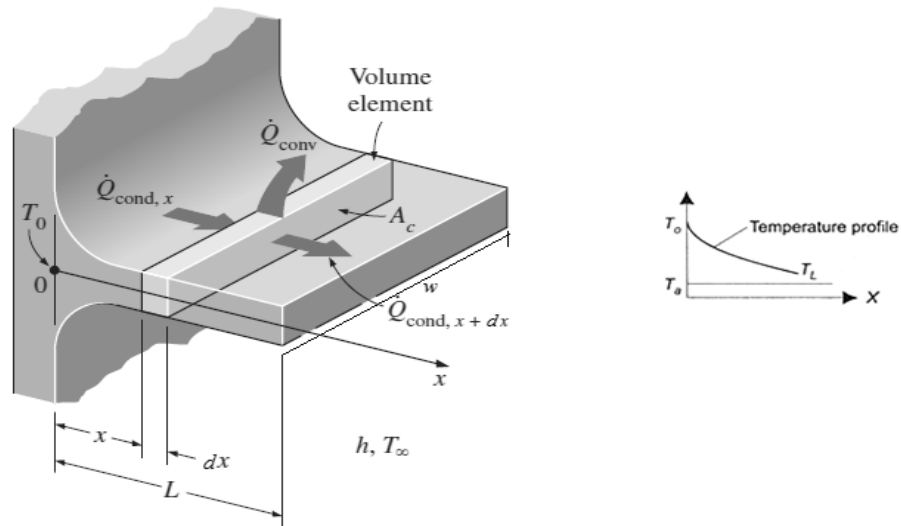
Consider an elemental section of thickness dx at a distance x from the base as shown. Let us write an energy balance for this element:

Energy going into the element by conduction = (Energy leaving the element by conduction) + Energy leaving the surface of the element by convection)

$$Q_x = Q_{x+dx} + Q_{conv} \dots \dots \dots a$$

Q_x = heat conducted into the element at x
Q_{x+dx} = heat conducted out of the element at x + dx, and
Q_{conv} = heat convected from the surface of the element to ambient

We have:

$$Q_x = -kA_c \frac{\partial T}{\partial x}$$

$$Q_{x+dx} = Q_x + \frac{\partial Q_x}{\partial x} dx$$

where: $\qquad Q_x = -kA_c \dfrac{\partial T}{\partial x} \qquad$ *Fourier's law sub. in last equation*

$$Q_{x+dx} = -kA_c \frac{\partial T}{\partial x} - kA_c \frac{\partial^2 T}{\partial x^2} dx$$

$$Q_{conv} = h A_s(T - T_a)$$

$$Q_{conv} = h (P \, dx)(T - T_a)$$

Where A_s is the surface area of the element, P is the perimeter substituting the terms in Eq. a:

$$-kA_c \frac{\partial T}{\partial x} = \left(-kA_c \frac{\partial T}{\partial x} - kA_c \frac{\partial^2 T}{\partial x^2} dx\right) + h\,(P\,dx)(T - T_a)$$

$$kA_c \frac{\partial^2 T}{\partial x^2} dx - h\,(P\,dx)(T - T_a) = 0$$

$$\frac{\partial^2 T}{\partial x^2} - m^2.(T - T_a) = 0 \dots\dots\dots\dots b$$

Where

$$m = \sqrt{\frac{h\,.\,P}{k\,.\,A_c}}$$

Note that m has units of: (m^{-1}) and is a constant, since for a given operating conditions of a fin, generally h and k are assumed to be constant.

Now, define excess temperature,

$$\theta = (T - T_a)$$

Therefore

$$\frac{\partial \theta}{\partial x} = \frac{\partial T}{\partial x} \quad and, \quad \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 T}{\partial x^2}$$

Substituting in Eq. b,

$$\frac{\partial^2 \theta}{\partial x^2} - m^2.\theta = 0 \dots\dots\dots.1$$

Eq.1 is the governing differential equation for the fin of uniform cross section considered.

Eq.1 is a second order, linear, ordinary differential equation. Its general solution is given by calculus theory, in two equivalent forms:

$$\theta(x) = C_1.e^{-mx} + C_2.e^{mx} \dots\dots\dots 2.a$$

where, C_1 and C_2 are constants
and,

$$\theta(x) = A.\cosh(mx) + B.\sinh(mx) \dots\dots\dots.2.b$$

where A and B are constants, and cosh and sinh are hyperbolic functions, defined in Table 1.

Eq.2a or .2b describes the temperature distribution in the fin along its length.

To calculate the set of constants C_1 and C_2, or A and B, we need two boundary conditions:

One of the B.C.'s is that the temperature of the fin at its base, i.e. at x = 0, is T_o and this is considered as known.

i.e. B.C. (1): at x = 0, T = T_o

Regarding the second boundary condition, there are several possibilities:

Case (1): Infinitely long fin.

Case (2): Fin insulated at its end (i.e. negligible heat loss from the end of the fin), and

Case (3): Fin losing heat from its end by convection.

It may be remarked here, that while for case (1), it is convenient to choose the solution in the form given by Eq. 2a and for cases (2) and (3), choosing the solution in the form given by Eq. 2b makes the analysis easy.

TABLE 1 Relations for hyperbolic functions

Sl. No.	Relation
(a)	$\cos h(\beta) = \dfrac{\exp(\beta) + \exp(-\beta)}{2}$
(b)	$\sin h(\beta) = \dfrac{\exp(\beta) - \exp(-\beta)}{2}$
(c)	$\exp(\beta) + \exp(-\beta) = 2\cos h(\beta)$
(d)	$\exp(\beta) - \exp(-\beta) = 2\sin h(\beta)$
(e)	$\exp(\beta) = \cos h(\beta) + \sin h(\beta)$
(f)	$\exp(-\beta) = \cos h(\beta) - \sin h(\beta)$
(g)	$\sin h(0) = 0$
(h)	$\cos h(0) = 1$
(i)	$\dfrac{d}{dx}(\sin h(m \cdot x) = m \cdot \cos h(m \cdot x)$
(j)	$\dfrac{d}{dx}\cos h(m \cdot x) = m \cdot \sin h(m \cdot x)$
(k)	$\cos h(-x) = \cos h(x)$
(l)	$\sin h(-x) = -\sin h(x)$
(m)	$\cos h(x + y) = \cos h(x) \cdot \cos h(y) + \sin h(x) \cdot \sin h(y)$
(n)	$\cos h(x - y) = \cos h(x) \cdot \cos h(y) - \sin h(x) \cdot \sin h(y)$
(o)	$\sin h(x + y) = \sin h(x) \cdot \cos h(y) + \cos h(x) \cdot \sin h(y)$
(p)	$\sin h(x - y) = \sin h(x) \cdot \cos h(y) - \cos h(x) \cdot \sin h(y)$

4.3 Case (1) Infinitely Long Fin

This simply means that the fin is very long. Consequence of this assumption is that temperature at the tip of the fin approaches that of the surrounding ambient as the fin length approaches infinity. See Fig.
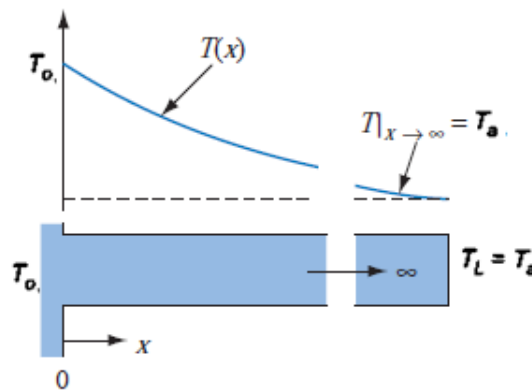
To determine the temperature distribution; The governing differential equation, as already derived, is given by Eq.1

$$\frac{\partial^2 \theta}{\partial x^2} - m^2 . \theta = 0 \; \dots\dots\dots 1$$

And, we shall choose for its solution for temperature distribution Eq. (2a), i.e.

$$\theta(x) = C_1 . e^{-mx} + C_2 . e^{mx} \; \dots\dots\dots 2.a$$

C1 and C2 are obtained from the B.C.s:

B.C. (1): $at\ x\ =\ 0, T\ =\ T_o$

B.C. (2): $at\ x \rightarrow \infty, T\ =\ T_a$, the ambient temperature.

From B.C. (1):

$$at\ x\ =\ 0, \qquad \theta(x) = T_o - T_a = \theta_o$$

From B.C. (2):

$$at\ x\ =\ \infty, \qquad \theta(x) = T_a - T_a = 0$$

From B.C. (2) and Eq. 2.a: $C_2 = 0$

From B.C. (1) and Eq. 2.a: $C_1 = \theta_o$

Substituting C_1 and C_2 back in Eq. 2.a, we get:

$$\theta(x) = \theta_o. e^{-mx}$$

$$\frac{\theta(x)}{\theta_o} = e^{-mx}$$

$$\frac{T(x) - T_a}{T_o - T_a} = e^{-mx} \ ....3$$

Eq. 3 gives the temperature distribution in an infinitely long fin of uniform cross section, along the length.

To determine the heat transfer rate:

Hat transfer rate from the fin may be determined by either of the two ways:

a) by the application of Fourier's law at the base of the fin, i.e. in steady state, the heat transfer from the fin must be equal to the heat conducted into the fin at its base.

$$Q_{fin} = -kA_c\frac{\partial T(x)}{\partial x} = -kA_c\frac{\partial \theta(x)}{\partial x}, \qquad at\ x = 0\ ......c$$

b) by integrating the convective heat transfer for the entire surface of the fin, i.e.

$$Q_{fin} = \int_0^L h.P.(T - T_a)dx = \int_0^L h.P.\theta dx\ ....d$$

By method (a):

$$Q_{fin} = -kA_c\frac{\partial T(x)}{\partial x} = -kA_c\frac{\partial \theta(x)}{\partial x}, \qquad at\ x = 0$$

$$Q_{fin} = -kA_c\frac{\partial \theta(x)}{\partial x}, \qquad at\ x = 0$$

$$Q_{fin} = -kA_c\left[\frac{\partial}{\partial x}(\theta_o.e^{-mx})\right], \qquad at\ x = 0$$

$$Q_{fin} = -kA_c(-m)[\theta_o.e^{-mx}], \qquad at\ x = 0$$

$$Q_{fin} = k.A_c.m.\theta_o\ ....4$$

Where

$$m = \sqrt{\frac{h.P}{k.A_c}}$$

$$Q_{fin} = \sqrt{h.P.k.A_c}.\theta_o = \sqrt{h.P.k.A_c}.(T_o - T_a)\ .....5$$

Eq. 4 and Eq. 5 gives the heat transfer rate through the fin.

Let us verify this result from method (b):

By method (a):

$$Q_{fin} = \int_0^L h.P.(T - T_a)dx = \int_0^L h.P.\theta dx \dots \dots \dots d$$

$$Q_{fin} = \int_0^\infty h.P.\theta_o.e^{-mx}dx$$

$$Q_{fin} = \frac{1}{m}.h.P.\theta_o$$

$$Q_{fin} = \sqrt{h.P.k.A_c}.\theta_o = \sqrt{h.P.k.A_c}.(T_o - T_a) \quad same\ Eq.5$$

4.4 Case (2) Fin of finite length with insulated end

End of a fin is generally not insulated, so here what we mean is that the heat transfer from the end of the fin is negligible as compared to the heat transfer from the surface of the fin. Mostly this is true, since the area of the end of the fin is negligible as compared to the exposed surface area of the fin; in fact this is the most important case. See fig.
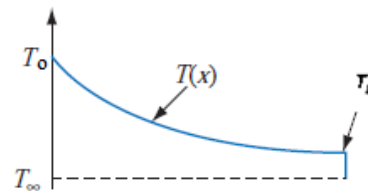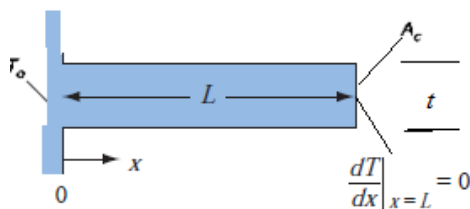
To determine the temperature distribution; The governing differential equation, as already derived, is given by Eq.1

$$\frac{\partial^2 \theta}{\partial x^2} - m^2.\theta = 0 \dots \dots \dots .1$$

And, we shall choose for its solution for temperature distribution Eq. (2b), i.e.

$$\theta(x) = A.\cosh(mx) + B.\sinh(mx) \dots \dots \dots .2.b$$

A and B are obtained from the B.C.s:

B.C. (1): $at\ x = 0, T = T_o$, so $\theta(x) = T_o - T_a = \theta_o$

B.C. (2): $at\ x = L$, $\frac{dT}{dX} = \frac{d\theta}{dX} = 0$ since the end is insulated.

From B.C. (1) and   Eq. 2b:

$$A = \theta_o$$

From B.C. (2) and   Eq. 2b:

$$\left(\frac{d\theta}{dX}\right)_{x=L} = 0$$

$$A.m.\sinh(mL) + B.m.\cosh(mL) = 0 \qquad using\ relations\ in\ table\ 1$$

Sub. A

$$\theta_o.m.\sinh(mL) + B.m.\cosh(mL) = 0$$

$$B = -\theta_o \frac{\sinh(mL)}{\cosh(mL)}$$

Substituting for A and B in Eq. 2b

$$\theta(x) = \theta_o . \cosh(mx) - \theta_o \frac{\sinh(mL)}{\cosh(mL)} . \sinh(mx)$$

$$\frac{\theta(x)}{\theta_o} = \frac{\cosh(mL) . \cosh(mx) - \sinh(mL) . \sinh(mx)}{\cosh(mL)}$$

$$\frac{\theta(x)}{\theta_o} = \frac{\cosh(m.(L-x))}{\cosh(mL)} \quad \text{... ... .6,} \quad using\ relation\ in\ table\ 1$$

$$\frac{T(x) - T_a}{T_o - T_a} = \frac{\cosh(m.(L-x)))}{\cosh(mL)} \quad \text{... ... .7}$$

Eq.6 or 7 gives the temperature distribution in the fin with negligible heat transfer from its end.
Temperature at the end of the fin:
This is easily determined by putting x=L in Eq. 6 or 7

$$\frac{\theta(L)}{\theta_o} = \frac{1}{\cosh(mL)} \quad \text{... ... ..6a}$$

$$\frac{T_L - T_a}{T_o - T_a} = \frac{1}{\cosh(m.L)} \quad \text{... ... .7a}$$

$$T_L = \frac{T_o - T_a}{\cosh(mL)} + T_a \quad \text{... ... 7b}$$

Eq. 7b gives the Temperature at the end of this fin (i.e. at x=L)

To determine the heat transfer rate:
Hat transfer rate from the fin may be determined by the application of Fourier's law at the base of the fin, i.e. in steady state, the heat transfer from the fin must be equal to the heat conducted into the fin at its base.

$$Q_{fin} = -kA_c \left(\frac{dT(x)}{dx}\right)_{x=0} = -kA_c \left(\frac{d\theta(x)}{dx}\right)_{x=0}$$

$$Q_{fin} = -kA_c\theta_o \left(\frac{-m.\sinh(m.(L-x))}{\cosh(m.L)}\right)_{x=0}$$

$$Q_{fin} = k.A_c.m.\theta_o . \tanh(m.L) \quad \text{... ... .8}$$

$$Q_{fin} = \sqrt{h.P.k.A_c}.\theta_o . \tanh(m.L) \quad \text{... ....9}$$

Eq.8 or 9 gives the heat transfer rate from the fin, insulated at its end.
Comparing Eq. 8 with that obtained for heat transfer from an infinitely long fin, i.e. Eq. 4, we see that a fin with insulated end becomes equivalent to an infinitely long fin $\tanh(m.L) = 1$.

4.5 Case (3) Fin of finite length losing heat from its end by convection

This is a more realistic case, though the relations developed are a little more complicated, see fig.
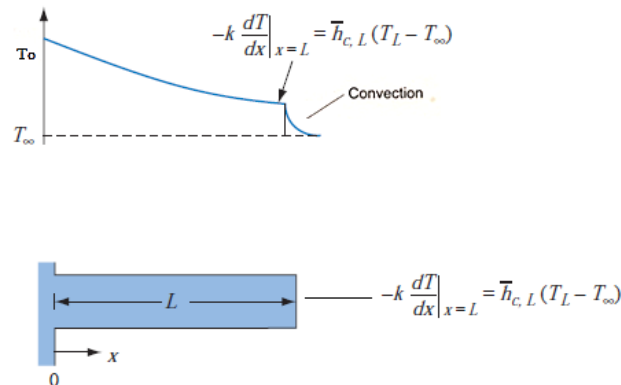
Here, heat conducted to the tip of the fin must be equal to the heat convected away from the tip to the ambient, i.e.

$$-kA_c\left(\frac{\partial T}{\partial x}\right)_{x=L} = h\,A_c(T_L - T_a)$$

$$-k.\left(\frac{\partial T}{\partial x}\right)_{x=L} = h.\,\theta_L$$

To determine the temperature distribution; The governing differential equation, as already derived, is given by Eq.1

$$\frac{\partial^2\theta}{\partial x^2} - m^2.\theta = 0 \ldots\ldots\ldots.1$$

And, we shall choose for its solution for temperature distribution Eq. (2b), i.e.

$$\theta(x) = A.\cosh(mx) + B.\sinh(mx) \ldots\ldots\ldots.2.b$$

A and B are obtained from the B.C.s:

B.C. (1): $at\ x = 0, T = T_o,\ so\ \theta(x) = T_o - T_a = \theta_o$

Applying B.C. (1) to Eq. 2b:

$$A = \theta_o$$

B.C. (2):

    heat conducted to the end = heat convected from the end

$$-kA_c\left(\frac{\partial\theta(x)}{\partial x}\right)_{x=L} = h\,A_c\theta(L), \qquad where\ \theta(L) = T_L - T_a$$

After applying the B.C.s and using Eq. 2b and the relation in table 1, we get:

$$\frac{\theta(x)}{\theta_o} = \frac{\cosh(m.(L-x)) + \dfrac{h}{m.k}\sinh(m(L-k))}{\cosh(m.L) + \dfrac{h}{m.k}\sinh(m.L)} \ldots\ldots.10$$

Eq. 10 gives the temperature distribution in a fin losing heat by convection at its end.

Note that when $h = 0$, i.e. for negligible heat transfer at the tip of the fin, Eq. 10 reduced to Eq. 6, for a fin with insulated tip.

To determine the heat transfer rate:

Hat transfer rate from the fin may be determined by the application of Fourier's law at the base of the fin, i.e. in steady state, the heat transfer from the fin must be equal to the heat conducted into the fin at its base.

$$Q_{fin} = -kA_c \left(\frac{dT(x)}{dx}\right)_{x=0} = -kA_c \left(\frac{d\theta(x)}{dx}\right)_{x=0}$$

$$Q_{fin} = kA_c\theta_o m \frac{\tanh(m.L) + \frac{h}{m.k}}{1 + \frac{h}{m.k}.\tanh(m.L)} \quad ......11$$

Eq. 11 gives the heat transfer rate from a fin losing heat by convection at its tip.

Note: Eq. 11 is important since it represents the heat transfer rate for a practically important case of a fin loosing heat from its end. However, it is rather complicated to use. So, in practice even when the fin loosing heat from its tip, it is easier to use Eq. 8 and 9 obtained for a fin with insulated tip, but with a correct length, L_c, rather than the actual length, L, to include the effect of convection at the tip. In that case only to evaluate Q, L is replaced by a corrected length L_c, in Eq. 8 and 9, as follows:

For rectangular fins: $\qquad L_c = L + \frac{t}{2}$ where t is the thickness of fin

For cylindrical (round) fins: $\qquad L_c = L + \frac{r}{2}$ where r is the radius of the cylindrical fin.

4.6 Performance of Fins

Recollect that purpose of attaching fins over a surface is to increase the heat transfer rate. How well this purpose is achieved is characterized by two performance parameters:

1) Fin efficiency, η_f, and
2) Fin effectiveness, ε_f.

4.6.1 Fin Efficiency

Fin transfers heat to the surroundings from its surface, by convection. For convection heat transfer, the driving force is the temperature difference between the surface and the surrounding. However, temperature drop along the length of the fin because of the finite thermal conductivity of the fin material; so, heat transfer becomes less effective towards the end of the fin. Obviously, in the ideal case of the entire fin being at the same temperature as that of the base wall, the heat transferred from the fin will be maximum, So fin efficiency is defined as the amount of heat actually transferred by a given fin to the ideal amount of heat that would be transferred if the entire fin were at its base temperature, i.e.

(a) Ideal

(b) Actual

$$\eta_f = \frac{Q_{fin}}{Q_{max}} \quad ......12$$

Where,

Q_{fin} = actual amount of heat transferred from the fin, and

Q_{max} = maximum (or ideal) amount of heat that would be transferred from the fin, if the entire fin surface were at the temperature of the base,
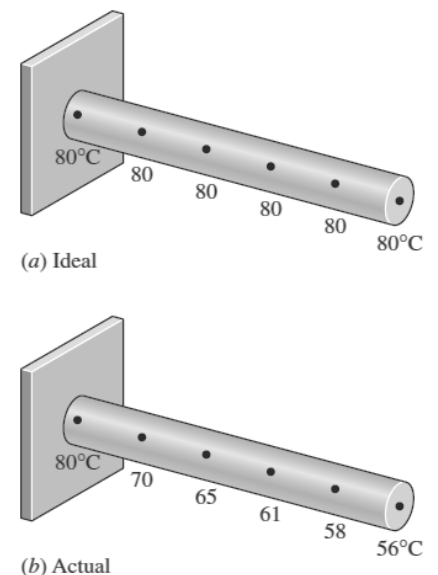
(a) For an infinitely long fin:

For an infinitely long fin, actual heat transferred is given by Eq. 5:

$$Q_{fin} = \sqrt{h.P.k.A_c}.\theta_o = \sqrt{h.P.k.A_c}.(T_o - T_a) \ldots..5$$

To calculate Q_{max}, if the entire fin surface were to be at a temperature of T_o, the convective heat transfer from the surface would be:

$$Q_{max} = h.P.L.(T_o - T_a) \ldots\ldots\ldots\ldots A$$

Where, P is the perimeter of the fin section and (P.L) is the surface area of the fin.
Dividing Eq. 5 by Eq. A:

$$\eta_f = \frac{\sqrt{h.P.k.A_c}.(T_o - T_a)}{h.P.L.(T_o - T_a)}$$

$$\eta_f = \frac{1}{\sqrt{\frac{h.P}{k.A_c}}.L}$$

$$\eta_f = \frac{1}{m.L} \ldots\ldots..13$$

(b) For a fin with insulated end:

For the case of a fin with an insulated end, we get actual heat transferred Q_{fin} from Eq. 9:

$$Q_{fin} = \sqrt{h.P.k.A_c}.\theta_o.\tanh(m.L) \ldots\ldots..9$$

and, fin efficiency is given by:

$$\eta_f = \frac{\sqrt{h.P.k.A_c}.(T_o - T_a).\tanh(m.L)}{h.P.L.(T_o - T_a)}$$

$$\eta_f = \frac{\tanh(m.L)}{\sqrt{\frac{h.P}{k.A_c}}.L}$$

$$\eta_f = \frac{\tanh(m.L)}{m.L} \ldots\ldots 14$$

$$\eta_f = \frac{\tanh(m.L)}{m.L}$$

Note For the more realistic case of a fin losing heat from its end, as stated earlier, to calculate heat transfer, Eq. 9 itself may be used, but, with a corrected length L_c in place of L.

Note: for a thin fin or a very wide fin i.e. w >> t, we can write:

$$m.L = \sqrt{\frac{h.P}{k.A_c}}.L = \sqrt{\frac{h.(2w+2t)}{k.w.t}}.L$$

$$m.L = \sqrt{\frac{2.h.w}{k.w.t}}.L = \sqrt{\frac{2.h}{k.t}}.L$$

Multiplying the numerator and the dominator by $L^{1/2}$ gives:

$$m.L = \sqrt{\frac{2.h}{k.t.L}}.L^{\frac{3}{2}}$$

$$m.L = \sqrt{\frac{2.h}{k.A_m}}.L^{\frac{3}{2}}$$

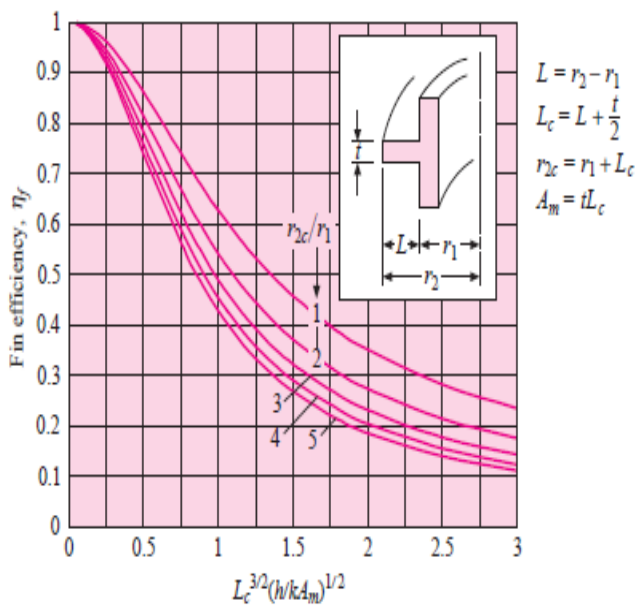Where, $A_m = (L.t)$, is the profile area for the rectangular section
Also

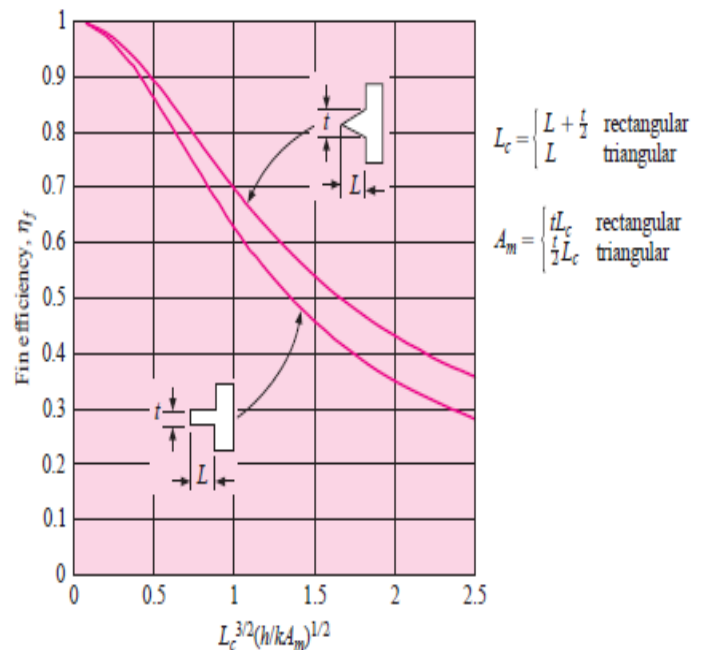$$\frac{P}{A_c} = \frac{(2w+2t)}{wt} = \frac{2w}{wt} = \frac{2}{t}$$

Also

$$m = \sqrt{\frac{h.P}{k.A_c}} = \sqrt{\frac{2.h}{k.t}}$$

Figure 1 | Efficiencies of circumferential fins of rectangular profile, according to Reference 3.


$L = r_2 - r_1$
$L_c = L + \frac{t}{2}$
$r_{2c} = r_1 + L_c$
$A_m = tL_c$

Figure 2 | Efficiencies of straight rectangular and triangular fins.

$L_c = \begin{cases} L + \frac{t}{2} & \text{rectangular} \\ L & \text{triangular} \end{cases}$

$A_m = \begin{cases} tL_c & \text{rectangular} \\ \frac{t}{2}L_c & \text{triangular} \end{cases}$

4.6.2 Fin Effectiveness (ε_f)

Consider a fin of uniform cross-sectional area A_c, fixed to a base surface. Purpose of the fin is to enhance the heat transfer. If the fin were not there, heat would have been transferred from the base area A_c, by convection. By attaching the fin, area for convection increases i.e. convective resistance ($1= /(h.A)$) decreases; however, it is obvious that a conduction resistance due to the solid fin is now introduced and the total heat transfer would depend upon the net thermal resistance. As we go on increasing the length of fin, convection resistance will go on decreasing, but conduction resistance will go on increasing. This means that attaching a fin may not necessarily result in effectively increasing the heat transfer. Therefore, how effective the fin is in enhancing the heat transfer is characterised by a parameter called fin effectiveness.
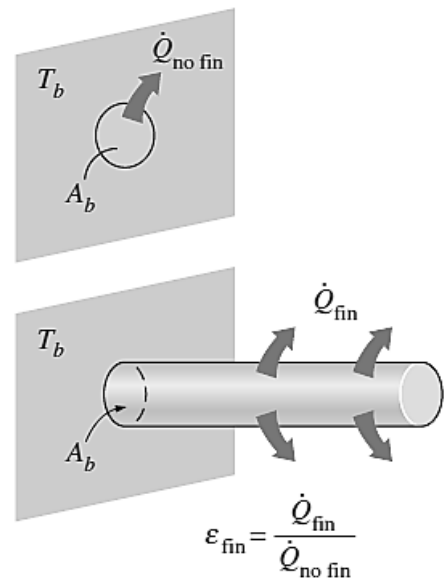
Fin effectiveness is defined as the ratio of the heat transfer rate with the fin in place, to the heat transfer that would occur if the fin were not there, from the area of the base surface where the fin was originally fixed.

ε_f = (heat transfer rate with fin)/(heat transfer rate without fin)

$$\varepsilon_f = \frac{Q_{fin}}{h.A_c.(T_o - T_a)} \dots\dots 16$$

Fin effectiveness equal to 1 means that there is no enhancement of heat transfer at all by using the; fin if the fin effectiveness is less than 1, that means that the fin actually reduces the heat transfer by adding additional thermal resistance! Obviously, ε_f should be as large as possible.

Use of fins is hardly justified unless fin effectiveness is greater than about 2, i.e. $\varepsilon_f > 2$.

To get an insight into the physical implications of fin effectiveness, let us consider an infinitely long fin

Then, we have:

$$\varepsilon_f = \frac{\sqrt{h.P.k.A_c}.(T_o - T_a)}{h.A_c.(T_o - T_a)}$$

$$\varepsilon_f = \sqrt{\frac{k.P}{h.A_c}} \dots\dots\dots 17$$

Eq. 17 is an important equation. Following significant conclusions may be derived from this equation:

1. Thermal conductivity, k should be as high as possible; that is why we see that generally, fins are made up of copper or aluminium. Of course, aluminium is the preferred material from cost and weight considerations.
2. Large ratio of perimeter to area of cross section is desirable; that means, thin, closely spaced fins are preferable. However, fins should not be too close as to impede the flow of fluid by convection.

3. Fins are justified when heat transfer coefficient h is small, i.e. generally on the gas side of a heat exchanger rather than on the liquid side. For example, the car radiator has fins on the outside of the tubes across which air flows.

4. Requirement that $\varepsilon_f > 2$, gives us the criterion:

$$\frac{k.P}{h.A_c} > 4 \ldots\ldots..18$$

These two important parameters, namely, η_f and ε_f are related to each other as follows:

$$\varepsilon_f = \frac{Q_{fin}}{Q_{base}} = \frac{Q_{fin}}{h.A_c.(T_o - T_a)} = \frac{\eta_f.h.A_f.(T_o - T_a)}{h.A_c.(T_o - T_a)}$$

$$\varepsilon_f = \frac{A_f}{A_c}\eta_f \ldots\ldots 19$$

-CHAPTER FIVE-

	UNSTEADY STATE CONDUCTION
'OR'	TRANSIENT HEAT CONDUCTION
'OR'	TIME DEPENDENT CONDUCTION

5.1 Introduction

5.2 Lumped-Heat-Capacity System

5.2.1 Criteria For Lumped System Analysis (Biot Number And Fourier Number)

5.3 One-Dimensional Transient Conduction In Large Body When Biot Number >0.1

5.3.1 Heisler Charts

5.4 Transient Heat Flow In A Semi-Infinite Solid

Unsteady State Conduction or Transient Heat Conduction or Time Dependent Conduction

5.1 Introduction

Earlier, we derived the general differential equation for conduction and then applied it to problems of increasing complexity, e.g. first we studied heat transfer in simple geometries without heat generation and then we studied heat transfer when there was internal heat generation. In all these problems, steady state heat transfer was assumed, i.e. the temperature within the solid was only a function of position and did not depend on time, i.e. mathematically, $T = T(x, y, z)$. However, all the process equipments used in engineering practice, such as boilers, heat exchangers, etc. have to pass through an unsteady state in the beginning when the process is started, and, they reach a steady state after sufficient time has elapsed. Or, as another example, a billet being quenched in an oil bath, goes through temperature variations with both position and time before it attains a steady state. Conduction heat transfer in such an unsteady state is known as transient heat conduction or, unsteady state conduction, or time dependent conduction. Obviously, in transient conduction, temperature depends not only on position in the solid, but also on time. So, mathematically, this can be written as $T = T(x, y, z, \tau)$, where τ represents the time coordinate.

Naturally, solutions for transient conduction problems are a little more complicated compared to steady state analysis, since now, an additional parameter, namely time (τ) is involved.

Typical examples of transient conduction occur in:

- (i) heat exchangers
- (ii) boiler tubes
- (iii) cooling of cylinder heads in I.C. engines
- (iv) heat treatment of engineering components and quenching of ingots
- (v) heating of electric irons
- (vi) heating and cooling of buildings
- (vii) freezing of foods, etc.

To solve a given one-dimensional, transient conduction problem, one could start with one of the relevant general differential equations discussed earlier and by solving it in conjunction with appropriate boundary conditions, and get the temperature distribution as a function of position and time.

5.2 Lumped-heat-capacity system

It is the systems that may be considered uniform in temperature. This type of analysis is called the *lumped-heat-capacity* method. If a hot steel ball were immersed in a cool pan of water, the lumped-heat-capacity method of analysis might be used if we could justify an assumption of uniform ball temperature during the cooling process. Clearly, the temperature distribution in the ball would depend on the thermal conductivity of the ball material and the heat-transfer conditions from the surface of the ball to the surrounding fluid (i.e., the surface-convection heat transfer coefficient).We should obtain a reasonably uniform temperature distribution in the ball if the resistance to heat transfer by conduction were small compared with the convection resistance at the surface, so that the major temperature gradient

would occur through the fluid layer at the surface. The lumped-heat-capacity analysis, then, is one that assumes that the internal resistance of the body is negligible in comparison with the external resistance. The convection heat loss from the body is evidenced as a decrease in the internal energy of the body, as shown in Figure (1). Thus

$$q = hA(T - T_a) = -\rho\, cp\, V\, \frac{dT}{d\tau} \,\ldots\ldots\ldots.1$$

$$h.A.(T - T_a).d\tau = -\rho.cp.V.dT$$

$$let \qquad \theta = (T - T_a), \qquad d\theta = dT$$

$$hA\theta = -\rho\, cp\, V\, \frac{d\theta}{d\tau}$$

$$\frac{d\theta}{\theta} = -\frac{h.A}{\rho.V.cp}.d\tau$$

Integration between $\tau{=}0$ (i.e. $\theta = \theta i$) to any τ $(i.e.\,\theta = \theta(\tau))$

$$\int_{\theta_i}^{\theta_\tau} \frac{d\theta}{\theta} = \int_0^\tau -\frac{h.A}{\rho.V.cp}.d\tau$$

$$ln\theta_\tau - ln\theta_i = -\frac{h.A}{\rho.V.cp}\tau$$

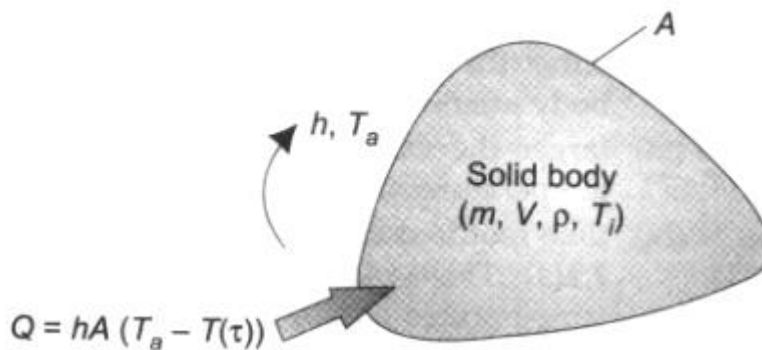$$ln\frac{\theta_\tau}{\theta_i} = \frac{-hA\tau}{\rho c_p V}$$

$$ln\left[\frac{(T(\tau) - Ta)}{(Ti - T_a)}\right] = \frac{-hA\tau}{\rho c_p V}$$

$$\frac{(T(\tau) - T_a)}{(Ti - T_a)} = exp\left[\frac{-hA\tau}{\rho c_p V}\right]\,\ldots\ldots..2$$

$\frac{\rho c_p V}{hA} = t$ its known as time constant and has a unit of time

$$\frac{(T(\tau) - T_a)}{(Ti - T_a)} = exp\left[\frac{-\tau}{t}\right]$$

Larger the value of time constant t, longer is the time required for the body to reach temperature

Figure(1) lumped system analysis

5.2.1 Criteria for lumped system analysis (Biot number and Fourier number)

For the simple analysis made above, we had the fundamental assumption that the internal conductive resistance of the body was negligible as compared to the convective resistance at its surface. This was stated in a rather qualitative way Now, let us study the criteria required for the lumped system analysis to be applicable. Consider a plane slab in steady state, transferring heat to a fluid on its surface with a heat transfer coefficient of h, as shown in the fig (2).
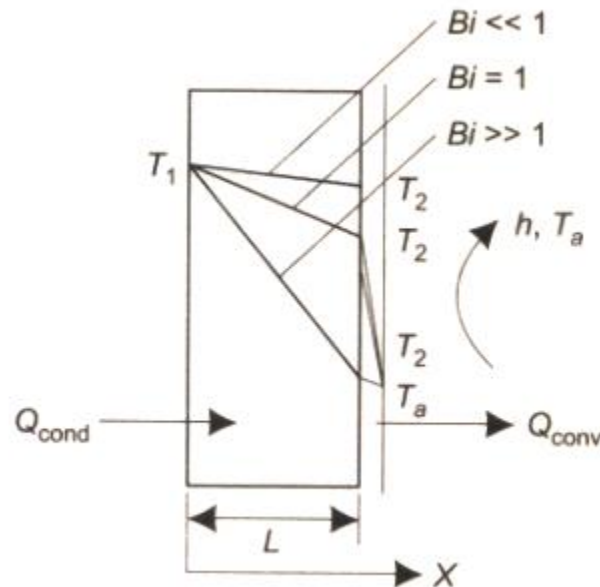

Figure (2) Biot number and temperature distribution in a plane wall

Let the surface on the left be maintained at temperature T_1 and the surface on the right is at a temperature of T_2 as a result of heat being   lost to a fluid at temperature T_a, flowing with a heat transfer coefficient h. Writing an energy balance at the right hand surface.

$$\frac{k.A}{L}.(T_1 - T_2) = h.A.(T_2 - T_a)$$

Rearranging

$$\frac{(T_1 - T_2)}{(T_2 - T_a)} = \frac{\frac{L}{kA}}{\frac{1}{hA}} = \frac{R_{cond}}{R_{conv}} = \frac{hL}{k} = Bi$$

the term, (h.L)/k, is a dimensionless number, known as <u>Biot number</u>. Biot number is a measure of the temperature drop in the solid relative to the temperature drop in the convective layer It is also interpreted as the ratio of conductive resistance in the solid to the convective resistance at its surface. Note from Fig.(2) the temperature profile for Bi « l It suggests that one can assume a uniform temperature distribution within the solid. For Bi « 1, temperature gradient in the solid is small and temperature can be taken as a function of time only. Note also that for Bi » 1, temperature drop across the solid is much larger than that across the convective layer at the surface. Therefore, lumped system analysis is applicable, only if Bi « l.

Therefore we can write eq.(2) as:

$$\frac{(T(\tau) - Ta)}{(Ti - T_a)} = exp\left[\frac{-hA\tau}{\rho c_p V}\right] \quad \text{if Bi} \ll 0.1$$

$$Bi = \frac{hL_c}{k}$$

Where $L_c = \frac{V}{A}$

L_c for common shapes:
1. plane wall =half thickness of the wall
2. cylinder = R/2
3. sphere = R/3
4. cube, side L = L/6

Now the term $\left[\frac{hA\tau}{\rho c_p V}\right]$ can be written as follows:

$$\left[\frac{hA\tau}{\rho c_p V}\right] = \left[\frac{hL_c}{k}\right] \cdot \left[\frac{k\tau}{\rho c_p L_c^2}\right] = \left[\frac{hL_c}{k}\right] \cdot \left[\frac{\alpha\tau}{L_c^2}\right] = Bi.Fo$$

$$where \qquad\qquad Fo = \left[\frac{\alpha\tau}{L_c^2}\right] = \text{Fourier number}$$

Fourier number: Like Biot number, is an important parameter in transient heat transfer problems. Its signifies the degree of penetration of heating or cooling effect through a solid. For small Fourier number large time will be required to get significant changes

 Now we can write eq. (2) as:

$$\frac{(T(\tau) - Ta)}{(Ti - T_a)} = \exp(-Bi.Fo) \qquad if\ Bi < 0.1$$

5.3 One-dimensional transient conduction in large plane walls, long cylinders and spheres when Biot number > 0.1

There are many situations in practice, where the temperature gradient in the solid is not negligible (i.e. Bi > 0.1) and the lumped system analysis is not applicable. In such situations, we start with the general differential equation for time dependent, one-dimensional conduction in the appropriate coordinate system and solve it in conjunction with the boundary conditions. In this section, we shall analyse one-dimensional transient conduction in large plane walls, long cylinders spheres when Bi > 0.1.
Fig. (3). shows schematic diagram and coordinate systems for a large, plane slab, long cylinder and a sphere.

at $\tau = 0$

(a) A large plane wall　　　(b) A long cylinder　　　(c) A sphere
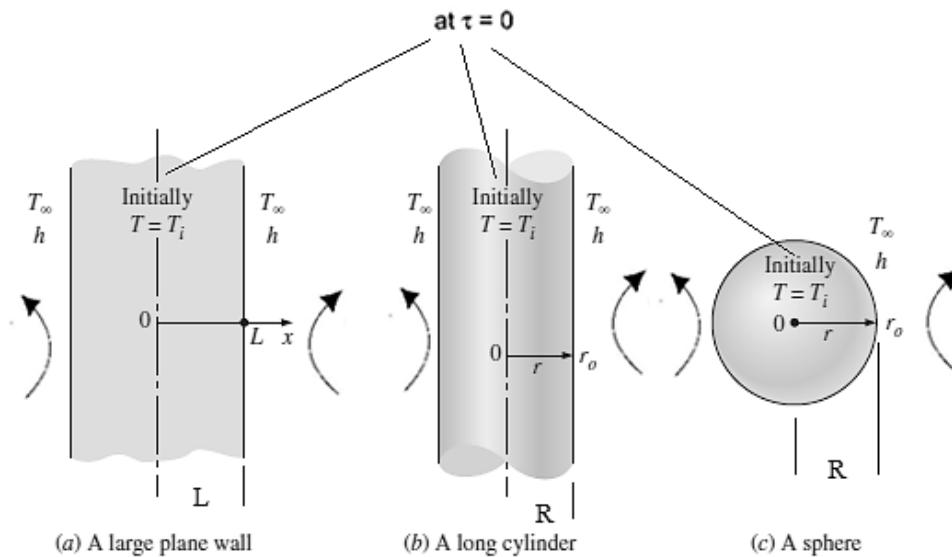
consider a plane slab of thickness 2L, shown in Fig. (a) above. Initially, i.e. at $\tau = 0$, the slab is at an uniform temperature, T_i, Suddenly, both the surfaces of the slab are subjected to convection heat transfer with an ambient at temperature T_a, with a heat transfer coefficient of h, as shown. Since there is geometrical and thermal symmetry, we need to consider only half the slab, and that is the reason why we chose the origin of the coordinate system on the mid-plane. Then, we can write the mathematical formulation of the problem for plane slab as follows:

$$\frac{d^2 T}{dx^2} = \frac{1}{\alpha} \frac{\partial T}{\partial \tau}$$

$$\frac{dT}{dx} = 0 \quad at\ x = 0, for\ \tau > 0$$

$$-k \frac{dT}{dx} = h.(T - T_a) \quad at\ x = L, for\ \tau > 0$$

$$T_i = T \quad for\ \tau = 0, in\ 0 < x < L$$

The solution of the above problem, however, is rather involved and consists of infinite series. So, it is more convenient to present the solution either in tabular form or charts.

Note: For all these three geometries, as mentioned, the solution involves infinite series, which are difficult to deal with. However, it is observed that for $Fo > 0.2$, considering only the first term of the series and neglecting other terms, involves an error of less than 2%. Generally, we are interested in times, Fo > 0.2. So, it becomes very useful and convenient to use one term approximation solution, for all these three cases

5.3.1 Heisler Charts
Results of analyses for these geometries have been presented in graphical form by Heisler in 1947. These graphs are shown in figs (5), (6) and (7) , for plane walls, long cylinders, and spheres, respectively.

How to use these charts?

First chart in each of these figures gives the non-dimensionalised centre temperature T_o. i.e. at x = 0 for the slab of thickness 2L, and at r = 0 for the cylinder and sphere, at a given time .

Temperature at any other position at the same time   , is calculated using the next graph, called 'position correction chart'. Third chart gives Q/Q_{max}.

Procedure of using these charts to solve a numerical problem is as follows:

1. First of all, calculate Bi from the data, with the definition of Bi, i.e. Bi = (h.L$_c$)/k, where L$_c$. is the characteristic dimension, given as: L$_c$ = (V/A) i.e. L$_c$ = L, half-thickness for a plane wall, L$_c$ = R/2 for a cylinder, and L$_c$=R/3 for a sphere. If Bi < 0.1, use lumped system analysis; otherwise, go for chart solution.

2. If Bi > 0.1, i.e. if we have to go for chart solution, calculate the Biot number again with the appropriate definition, i.e. $Bi = (hL/k)$ for a plane wall where L is half-thickness, and (hR/k) for a cylinder or sphere, where R is the outer radius. Also, calculate Fourier number, $Fo = \alpha.\tau/L^2$ for the plane wall, and $Fo = \alpha.\tau/R^2$ for a cylinder or sphere.

3. To calculate the centre temperature, use chart (a) from Figs. (5), (6) and (7), depending upon the geometry being considered. Enter the chart on the x-axis with the calculated Fo and draw a vertical line to intersect the $(1/Bi)$ line; from the point of intersection, move horizontally to the left to the y-axis to read the value of $\theta_o = (T_o - T_a)/(T_i - T_a)$ Here, T$_o$ is the centre temperature, which can now be calculated since T$_i$ and T$_o$ are known.

4. To calculate the temperature at any other position, use Fig. (b) of Figs. (5), (6) and (7), as appropriate. Enter the chart with $1/Bi$ on the x-axis, move vertically up to intersect the (x/L) or (r/R) curve as the case may be, and from the point of intersection, move to the left to read on the y-axis, the value of $\theta = (T - T_a)/(T_o - T_a)$. Here, T is the desired temperature at the indicated position.

5. To find out the amount of heat transferred Q, during a particular time interval τ from the beginning (i.e. $\tau = 0$), use Fig. (c) from Figs. (5), (6) and (7), depending upon the geometry. Enter the x-axis with the value of (Bi^2.Fo) and move vertically up to intersect the curve representing the appropriate Bi and move to the left to read on the y-axis, the value of Q/Q$_{max}$, Q is then easily found out since $Q_{max} = m\ Cp\ (T_i - T_a)$.

Note the following in connection with these charts:

1- These charts are valid for Fourier number Fo > 0.2.

2- Specifically, remember that while calculating Biot number, characteristic length (Lc) used is L, the half-thickness for a plane wall, and outer radius, R for the cylinder and the sphere (Lc is, now, not equal to:(V/A)).

3- Note from the 'position correction charts' that at $Bi < 0.1\ (i.e.\ 1/Bi > 10)$, temperature within the body can be taken as uniform, without introducing an error of more than 5%. This was precisely the condition for application of lumped system analysis'.

4- It is difficult to read these charts accurately, since logarithmic scales are involved; also the graphs are rather crowded with lines. However, these graphs are extremely useful for a quick estimation of values required.

5.4 Transient Heat Flow In A Semi-Infinite Solid

See Fig. 4. The solid is initially at a uniform temperature T_i and for times $\tau > 0$, the boundary surface at x= 0 is maintained at temperature T_s. Starting with the differential Eq. for one-dimensional, time dependent   conduction, for these boundary conditions, the non-dimensional temperature distribution (solved by the Laplace-transform technique) in the solid is obtained

as:

$$\frac{T(x,\tau) - Ts}{Ti - Ts} = erf\frac{x}{2\sqrt{\alpha\tau}} \dots \dots \dots \dots \dots \dots .3$$

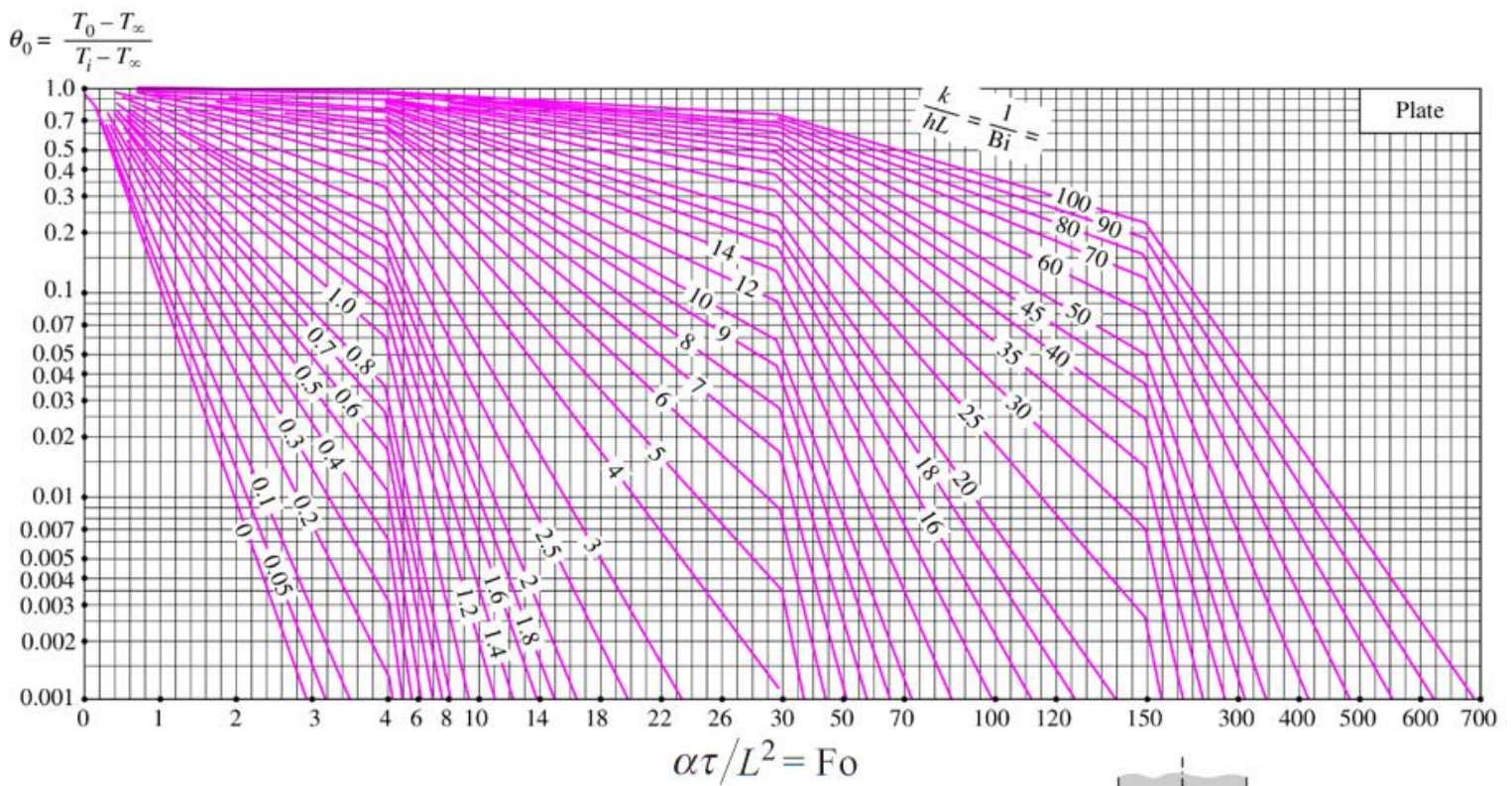where the Gauss error function $\{(erf(\zeta)\}$ is tabulated in table 1.
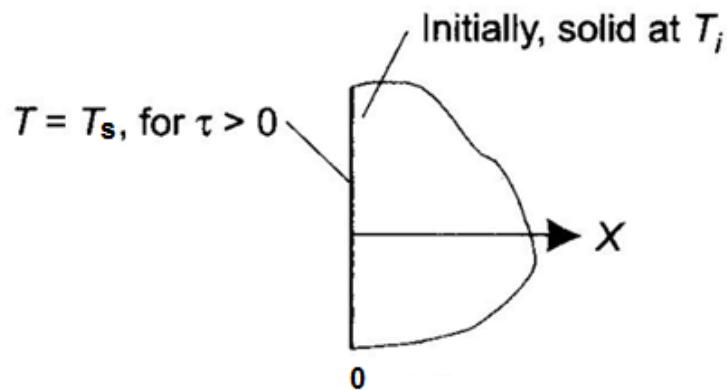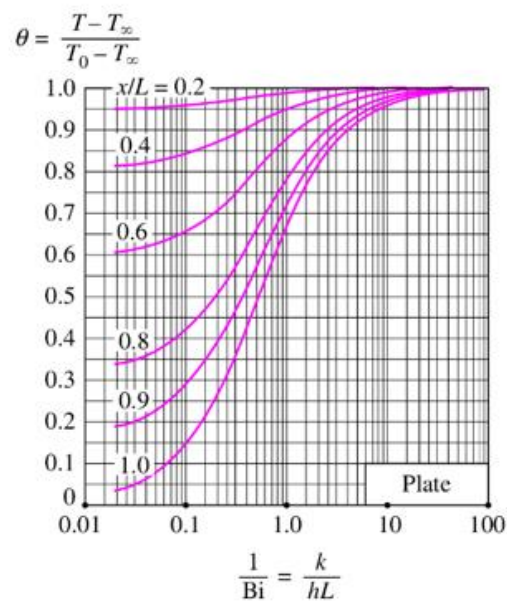

Figure (4)

so: $\qquad \zeta = \dfrac{x}{2\sqrt{\alpha\tau}} \qquad$ and, $\quad erf(\zeta) = \dfrac{T(x,\tau) - Ts}{Ti - Ts}$

Total heat flow during $\tau = 0$ to $\tau = \tau$

$$Q_{total} = 1.13.\,k.\,A(To - Ti).\sqrt{\frac{\tau}{\alpha}} \qquad J$$

$$\theta_0 = \frac{T_0 - T_\infty}{T_i - T_\infty}$$

$$\alpha\tau/L^2 = \text{Fo}$$

(a) Midplane temperature (from M. P. Heisler, "Temperature Charts for Induction and Constant Temperature Heating," *Trans. ASME 69*, 1947, pp. 227–36. Reprinted by permission of ASME International.)

$$\theta = \frac{T - T_\infty}{T_0 - T_\infty}$$

$$\frac{1}{\text{Bi}} = \frac{k}{hL}$$

$$\frac{Q}{Q_{max}}$$

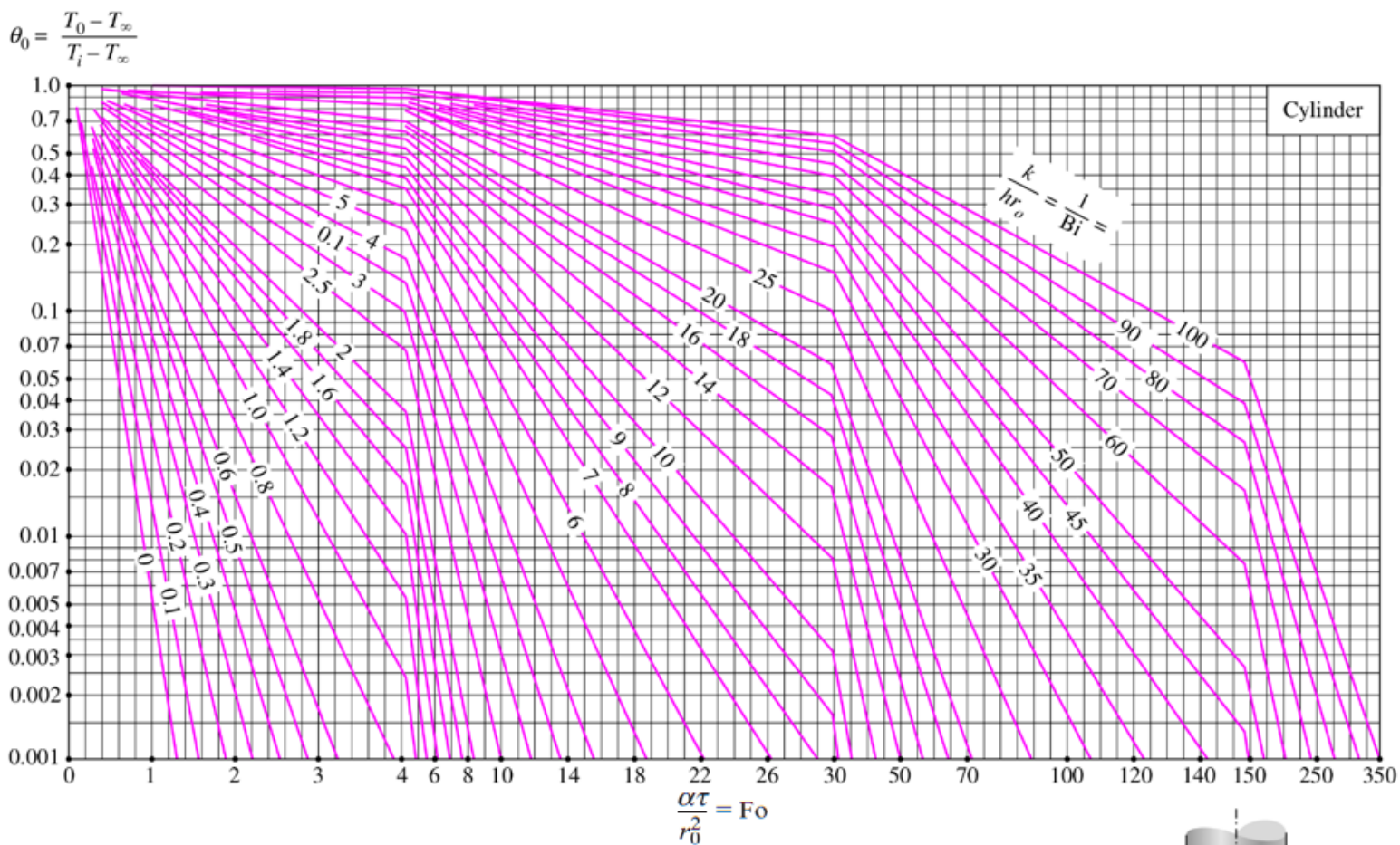$$\frac{h^2\alpha\tau}{k^2} = \text{Fo Bi}^2$$

(b) Temperature distribution (from M. P. Heisler, "Temperature Charts for Induction and Constant Temperature Heating," *Trans. ASME 69*, 1947, pp. 227–36. Reprinted by permission of ASME International.)
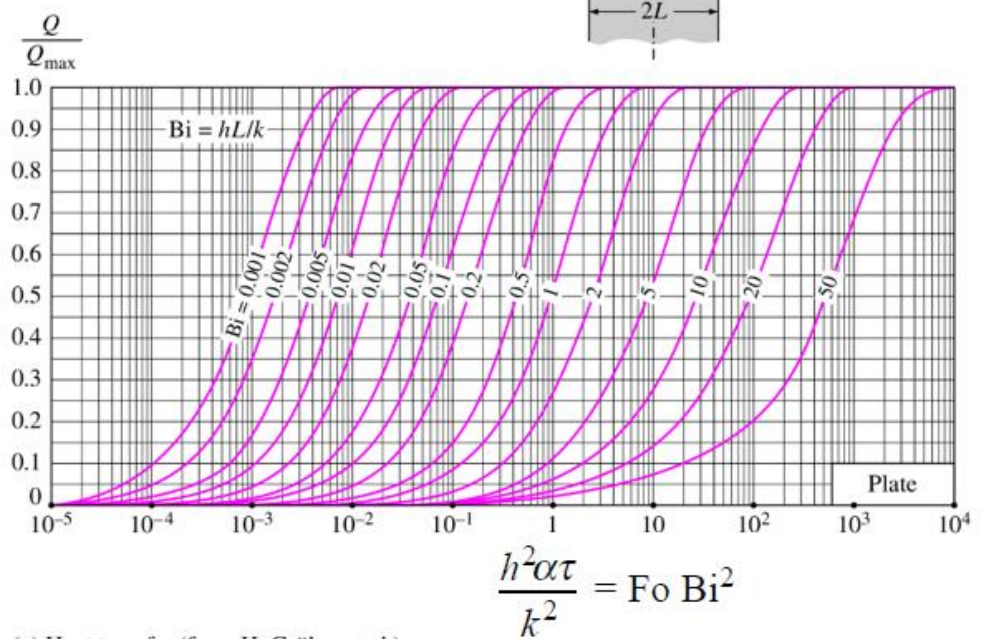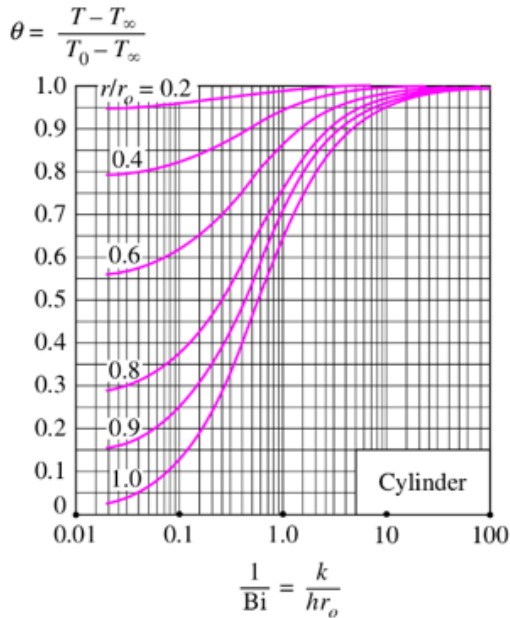
(c) Heat transfer (from H. Gröber et al.)

Figure (5)

$\theta_0 = \dfrac{T_0 - T_\infty}{T_i - T_\infty}$

$\dfrac{\alpha \tau}{r_0^2} = \text{Fo}$

(a) Centerline temperature (from M. P. Heisler, "Temperature Charts for Induction and Constant Temperature Heating," *Trans. ASME 69*, 1947, pp. 227–36. Reprinted by permission of ASME International.)
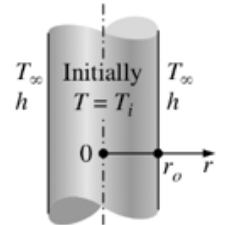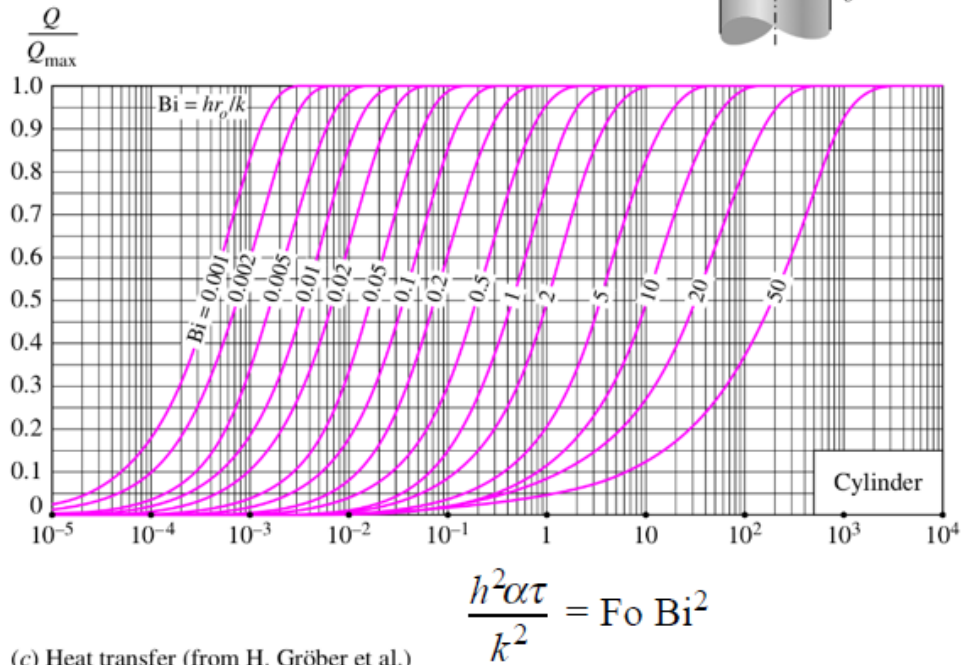
$\theta = \dfrac{T - T_\infty}{T_0 - T_\infty}$

$\dfrac{1}{\text{Bi}} = \dfrac{k}{hr_o}$

$\dfrac{Q}{Q_{max}}$
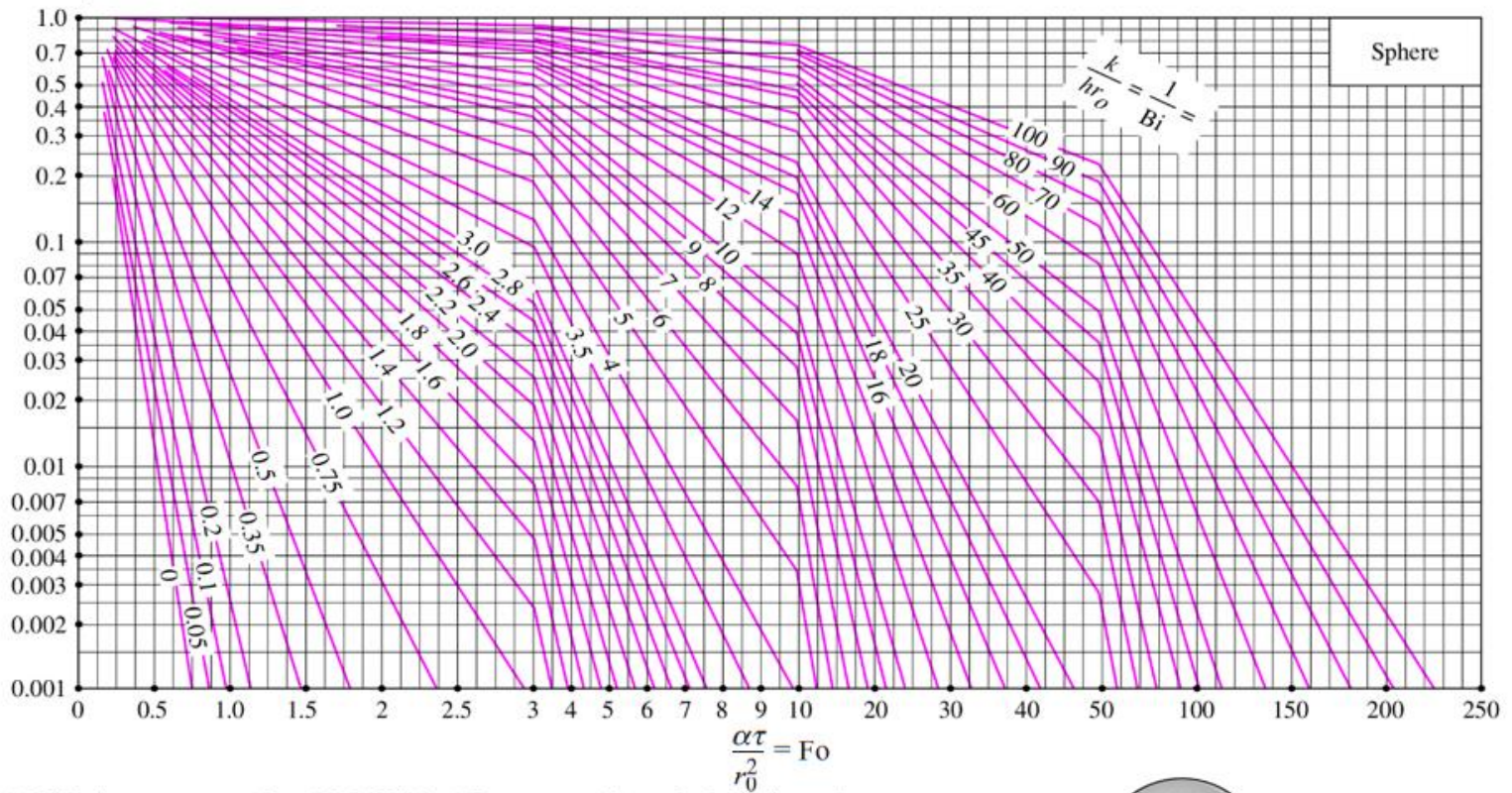

$\dfrac{h^2 \alpha \tau}{k^2} = \text{Fo Bi}^2$

(b) Temperature distribution (from M. P. Heisler, "Temperature Charts for Induction and Constant Temperature Heating," *Trans. ASME 69*, 1947, pp. 227–36. Reprinted by permission of ASME International.)

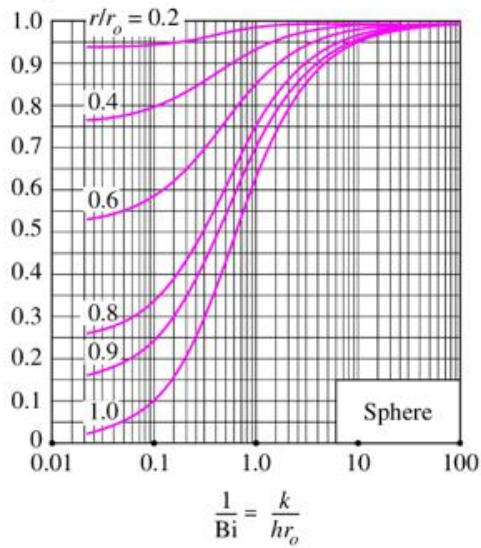(c) Heat transfer (from H. Gröber et al.)

Figure (6)

$$\theta_0 = \frac{T_0 - T_\infty}{T_i - T_\infty}$$

(a) Midpoint temperature (from M. P. Heisler, "Temperature Charts for Induction and Constant Temperature Heating," *Trans. ASME 69*, 1947, pp. 227–36. Reprinted by permission of ASME International.)

$$\theta = \frac{T - T_\infty}{T_0 - T_\infty}$$

(b) Temperature distribution (from M. P. Heisler, "Temperature Charts for Induction and Constant Temperature Heating," *Trans. ASME 69*, 1947,

(c) Heat transfer (from H. Gröber et al.)

$$\frac{h^2 \alpha \tau}{k^2} = \text{Fo Bi}^2$$
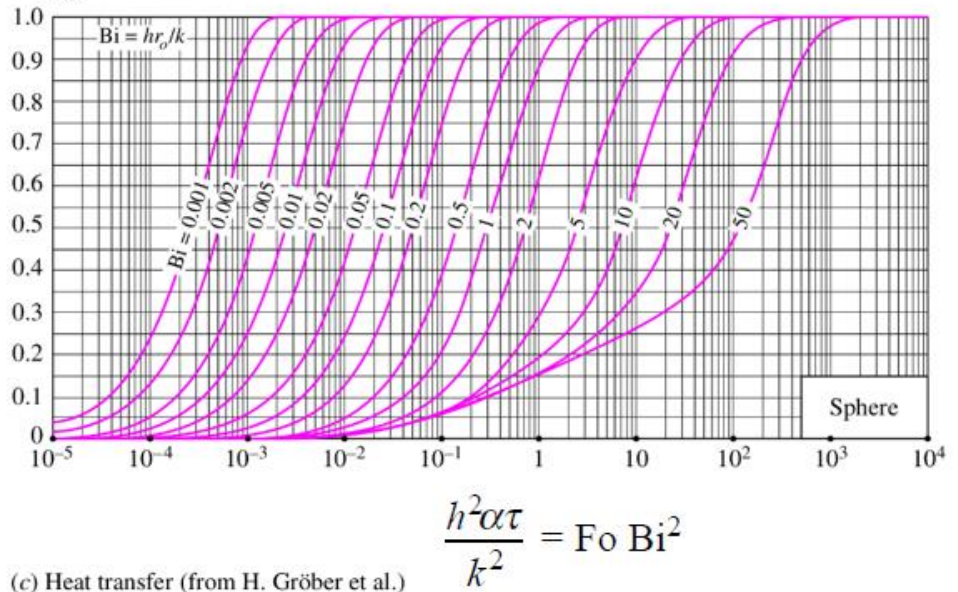
Figure (7)

Table 1 | The error function.

$\dfrac{x}{2\sqrt{\alpha\tau}}$	$\mathrm{erf}\,\dfrac{x}{2\sqrt{\alpha\tau}}$	$\dfrac{x}{2\sqrt{\alpha\tau}}$	$\mathrm{erf}\,\dfrac{x}{2\sqrt{\alpha\tau}}$	$\dfrac{x}{2\sqrt{\alpha\tau}}$	$\mathrm{erf}\,\dfrac{x}{2\sqrt{\alpha\tau}}$
0.00	0.00000	0.76	0.71754	1.52	0.96841
0.02	0.02256	0.78	0.73001	1.54	0.97059
0.04	0.04511	0.80	0.74210	1.56	0.97263
0.06	0.06762	0.82	0.75381	1.58	0.97455
0.08	0.09008	0.84	0.76514	1.60	0.97636
0.10	0.11246	0.86	0.77610	1.62	0.97804
0.12	0.13476	0.88	0.78669	1.64	0.97962
0.14	0.15695	0.90	0.79691	1.66	0.98110
0.16	0.17901	0.92	0.80677	1.68	0.98249
0.18	0.20094	0.94	0.81627	1.70	0.98379
0.20	0.22270	0.96	0.82542	1.72	0.98500
0.22	0.24430	0.98	0.83423	1.74	0.98613
0.24	0.26570	1.00	0.84270	1.76	0.98719
0.26	0.28690	1.02	0.85084	1.78	0.98817
0.28	0.30788	1.04	0.85865	1.80	0.98909
0.30	0.32863	1.06	0.86614	1.82	0.98994
0.32	0.34913	1.08	0.87333	1.84	0.99074
0.34	0.36936	1.10	0.88020	1.86	0.99147
0.36	0.38933	1.12	0.88079	1.88	0.99216
0.38	0.40901	1.14	0.89308	1.90	0.99279
0.40	0.42839	1.16	0.89910	1.92	0.99338
0.42	0.44749	1.18	0.90484	1.94	0.99392
0.44	0.46622	1.20	0.91031	1.96	0.99443
0.46	0.48466	1.22	0.91553	1.98	0.99489
0.48	0.50275	1.24	0.92050	2.00	0.995322
0.50	0.52050	1.26	0.92524	2.10	0.997020
0.52	0.53790	1.28	0.92973	2.20	0.998137
0.54	0.55494	1.30	0.93401	2.30	0.998857
0.56	0.57162	1.32	0.93806	2.40	0.999311
0.58	0.58792	1.34	0.94191	2.50	0.999593
0.60	0.60386	1.36	0.94556	2.60	0.999764
0.62	0.61941	1.38	0.94902	2.70	0.999866
0.64	0.63459	1.40	0.95228	2.80	0.999925
0.66	0.64938	1.42	0.95538	2.90	0.999959
0.68	0.66278	1.44	0.95830	3.00	0.999978
0.70	0.67780	1.46	0.96105	3.20	0.999994
0.72	0.69143	1.48	0.96365	3.40	0.999998
0.74	0.70468	1.50	0.96610	3.60	1.000000

-CHAPTER SIX-

PRINCIPLE OF CONVECTION AND BOUNDARY LAYER

6.1 Introduction

6.1.1 Newton's Law of Cooling and Heat Transfer Coefficient

6.1.2 Nusselt Number

6.2 Velocity boundary layer

6.3 Thermal Boundary Layer

6.4 Differential Equations for the Boundary Layer

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6.5 Relation between the fluid friction and heat transfer coefficient for a flat plate.

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6.6.1   Turbulent Boundary Layer Flow Over A Flat Plate

6.6.2 Flow across Tube Banks

6.7 Empirical and Practical Relations for Internal Flow

6.7.1   Flow Inside Tubes

6.7.2   Helically Coiled Tubes

PRINCIPLE OF CONVECTION AND BOUNDARY LAYER

6.1 Introduction

In the previous lectures, we studied about conduction heat transfer, where heat transfer was a molecular phenomenon and was considered mainly in solids; convection was mentioned only in passing and was considered only as a boundary condition while analysing conduction heat transfer.

In convection heat transfer, there is a flow of fluid associated with heat transfer and the energy transfer mainly due to bulk motion of the fluid. When the flow of fluid is caused by an external agency such as a fan or pump, the resulting heat transfer is known as 'Forced convection heat transfer; when the flow of fluid is due to density differences caused by temperature differences, the heat transfer is said to be by 'Natural (or free) convection'. For example, if air is blown on a hot plate by a blower, heat transfer occurs by forced convection, whereas, a hot plate simply hung in air will lose heat by natural convection.

6.1.1 Newton's Law of Cooling and Heat Transfer Coefficient

Governing rate equation for convection heat transfer is given by 'Newton's Law of Cooling. According to this law, the heat flux in convection heat transfer is given by.:

$$q_{conv} = h.(T_s - T_a) \dots . a$$

where h is the convective heat transfer coefficient and $(T_s - T_a)$ is the temperature difference between the hot surface and the flowing fluid. Unit of h is: $W/(m^2 C)$ so that the heat flux has units of W/m^2.

Though Eq. a looks very simple, it is very subtle. The reason is: heat transfer coefficient, h, depend on several factors such as:

1. the fluid properties like density, viscosity, thermal conductivity and specific heat,
2. type of flow (laminar or turbulent) i.e. (velocity of the flow, shape of fluid passage(circular, rectangle or a flat surface)),
3. nature of the surface (rough/smooth) and
4. orientation of the surface

In fact, entire thrust in determining the heat transfer rate in convection is to find out this value of "h" in a reliable manner.

6.1.2 Nusselt Number

Since we know that adjacent to the solid surface the fluid layer is stationary and the heat transfer in this fluid layer is by conduction

$$q_{cond} = -k_f(dT/dy) \dots . b$$

and the heat transferred by convection subsequently must be equal to this fluid layer, we can equate Eqs. a and b:

$$h.(T_s - T_a) = -k_f(dT/dy)$$

$$h = (-k_f(dT/dy))/(T_s - T_a)$$

i.e. the problem of finding out the value of "h" reduces to the task of finding out the temperature gradient (dT/dy) at y = 0 i.e. at the surface. Since the heat transfer coefficient depends on flow conditions, its value on a surface varies from point to point. However, we

generally take an average value of "*h*" by properly averaging the local value of heat transfer coefficient over the entire surface.

It is also common practice to non-dimensionalise the heat transfer coefficient with 'Nusselt number. Nusselt number is defined as:

$$Nu = \frac{h.\delta}{k_f}$$

where δ is a characteristic dimension and k_f is the fluid thermal conductivity.

To get a physical interpretation of the Nusselt number, consider a thin layer of fluid with thickness δ and with a temperature difference of ΔT between the two surfaces. Then, we have:

$$\frac{q_{conv}}{q_{cond}} = \frac{h.\Delta T}{k_f . \frac{\Delta T}{\delta}} = \frac{h.\delta}{k_f} = Nu$$

In other words, Nusselt number tells us how much the heat transfer is enhanced due to convection as compared to only conduction. Or, higher the Nusselt number, larger the heat transfer by convection.

6.2 Velocity boundary layer

Let us first study the development of boundary layer for a flow over a flat plate. Flow over a flat plate is important from a practical point of view, since flow over turbine blades and aerofoil sections of air plane wings can be approximated as flow over a flat plate. See Fig. 1.

Consider a thin, flat plate. The leading edge and the trailing edge of the plate are shown in the Fig.1. Let a fluid approach the flat plate at a free stream velocity of U. The fluid layer immediately in contact with the plate surface adheres to the surface and remains stationary, and in fluid mechanics, this phenomenon is known as 'no slip condition. Then, the fluid layer next to this stationary layer has its velocity retarded because of the viscosity effects i.e. due to the frictional force or 'drag' exerted between the stationary and the moving layers. This effect continues with subsequent layers up to some distance in the y-direction till the velocity equals the free stream velocity U. This region of fluid layer in which the viscosity effects are predominant is known as the 'velocity (or hydrodynamic) boundary layer', or simply the 'boundary layer'.
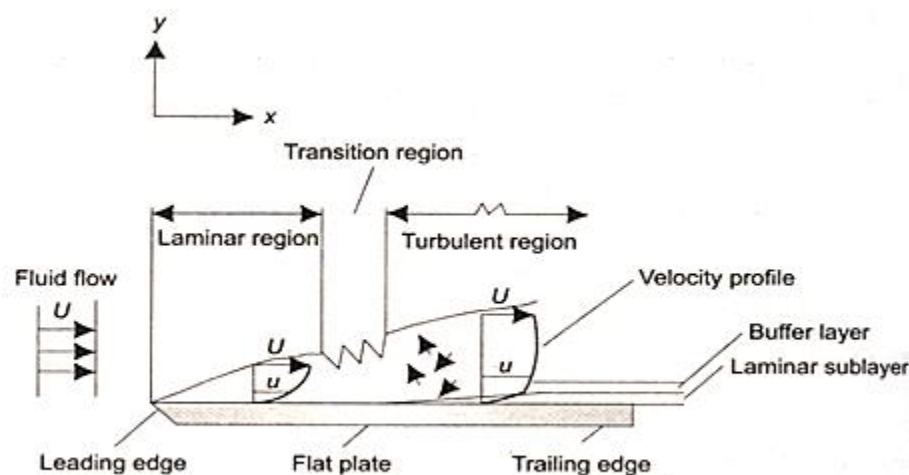

FIGURE 1   Development of boundary layer over a flat plate

Note the following points in connection with the boundary layer:

1. The boundary layer divides the flow field into two regions: one, 'the boundary layer region' where the viscosity effects are predominant and the velocity gradients are very steep, and, second, inviscid region' where the frictional effects are negligible and the velocity remains essentially constant at the free stream value.
2. Since the fluid layers in the boundary layer travel at different velocities, the faster layer exerts a drag force (or frictional force) on the slower layer below it; the drag force per unit area is known as shear stress (τ)'. Shear stress is proportional to the velocity gradient at the surface. This is the reason why in fluid mechanics, the velocity profile has to be found out to determine the frictional force exerted by a fluid on the surface. Shear stress is given by:

$$\tau_s = \mu.\left(\frac{dU}{dy}\right)_{y=0} \qquad N/m^2$$

Where μ is 'dynamic viscosity' of the fluid; its unit is kg/(ms). Viscosity is a measure of resistance to flow. For liquids, viscosity decreases as temperature increases.

3. Use of the above Eq. to determine the surface shear stress is not very convenient, since it requires a mathematical expression for the velocity profile; so, in practice, surface shear stress is determined in terms of the free stream velocity from the following relation:

$$\tau_s = C_f \frac{\rho U^2}{2} \qquad N/m^2$$

where C_f is a 'friction coefficient' or 'drag coefficient', ρ is the density of the fluid. C_f is determined experimentally in most cases. Drag coefficient varies along the length of the flat plate. Average value of drag coefficient (C_{fa}) is calculated by suitably integrating the local value over the whole length of the plate and then the drag force over the entire plate surface is determined from:

$$F_D = C_{fa}.A.\frac{\rho U^2}{2} \qquad N$$

where A = surface area, m^2.

4. Starting from the leading edge of the plate, for some distance along the length of the plate, the flow in the boundary layer is 'laminar' i.e. the layers of fluid are parallel to each other and the flow proceeds in a systematic, orderly manner. However, after some distance, disturbances appear in the flow and beyond this 'transition region', flow becomes completely chaotic and there is complete mixing of 'chunks' of fluid moving in a random manner i.e. the flow becomes 'turbulent'.
5. Transition from laminar to turbulent flow depends primarily on the free stream velocity, fluid properties, surface temperature and surface roughness, and is characterized by 'Reynolds number'. Reynolds number is a dimensionless number, defined as:
Re = (Inertia forces/Viscous forces). Or,

$$Re = \frac{U.x}{v}$$

where

U = free stream velocity, m/s

x = characteristic length i.e. for a flat plate it is the length along the plate in the flow direction, from the leading edge, and

v = kinematic viscosity of fluid = μ/ρ , m^2/s, where ρ is the density of fluid.

When the Reynolds number is low, i.e. when the flow is laminar, inertia forces are small compared to viscous forces and the velocity fluctuations are 'damped out' by the viscosity effects and the layers of fluid flow systematically, parallel to each other. When the Reynolds number is large, i.e. when the flow is turbulent, inertia forces are large compared to the viscous forces and the flow becomes chaotic. For a flat plate, in general, for practical purposes, the 'critical Reynolds number, at which the flow changes from laminar to turbulent is taken as $(5 * 10^5)$. It should be understood clearly that this is not a fixed value but depends on many parameters including the surface roughness.

6. There is intense mixing of fluid particles in turbulent region; therefore, heat transfer is more in turbulent flow as compared in laminar flow. This is the reason why special efforts are made in the design of heat exchangers to increase turbulence. However, one has to pay a premium of increased pressure drop i.e. increased power to pump the fluid through the heat exchanger.

7. Turbulent boundary layer itself is made of three layers: a very thin layer called laminar sub-layer', then, a "buffer layer' and, finally, the 'turbulent layer',

8. Thickness of the boundary layer, δ, increases along the flow direction; δ is related to the Reynolds number as follows: in the laminar flow region:

$$\delta_{lam} = \frac{5.x}{(Re_x)^{0.5}}$$

and for turbulent flow region:

$$\delta_{turb} = \frac{0.376.x}{(Re_x)^{0.2}}$$

where Re_x is the Reynolds number at position x from the leading edge.

6.3 Thermal Boundary Layer

When the temperature of a fluid flowing on a surface is different from that of the surface, a 'thermal boundary layer' develops on the surface, in a manner similar to the development of the velocity boundary layer. Let us illustrate the development of the thermal boundary layer with reference to a flat plate. See Fig. 2.

Consider a fluid at an uniform velocity of U and a uniform temperature of T_a approach the leading edge of a thin, flat plate as shown. Let the flat plate be at a uniform temperature of T_s. Let $T_a > T_s$. Then, the first layer that comes in contact with the surface will adhere to the surface (no slip condition) and reach thermal equilibrium with the surface and attain a temperature of T_s Then, the fluid particles in this layer will exchange energy with the particles in the adjoining layer, which in turn will exchange energy with the subsequent layer, and so on. Thus a temperature profile will develop in the flow field and the temperature will vary from T_s at the

surface to T_a at the free stream. The region in which the temperature variation in the y-direction is significant is known as 'thermal boundary layer'. Thickness of the thermal boundary Layer at any location is defined as that distance from the plate surface in the y-direction where the temperature difference between the fluid and the surface has reached 99% of the maximum possible temperature difference of (T_a - T_s).
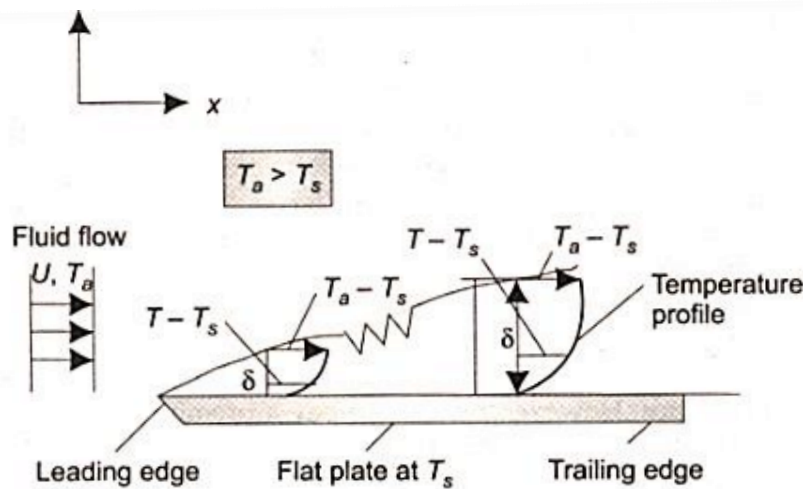
FIGURE 2   Development of thermal boundary layer over a flat plate

velocity profile in the hydrodynamic boundary layer depends on the viscosity of the fluid, whereas temperature profile in the thermal boundary layer depends on the viscosity, specific heat and thermal conductivity of the fluid, in addition to the velocity.

Relative magnitudes of the thicknesses of the hydrodynamic boundary layer (δ) and thermal boundary layer (δ_t) depend on the dimensionless parameter 'Prandtl number' defined as:

$pr = (Molecular\ diffusivity\ of\ momentum)/(Molecular\ diffusivity\ of\ heat)$

Or,

$$pr = \frac{v}{\alpha} = \frac{\mu.C_p}{k}$$

Where μ is dynamic viscosity, C_p is the specific heat and k is the thermal conductivity of the fluid.

Also, v is kinematic viscosity =   , and α is the thermal diffusivity.

Prandtl number is of the order of 1 for gases, less than 0.01 for liquid metals and more than 1000 for heavy oils.

Regarding the relative growth of velocity and thermal boundary layers in a fluid, we may note the following:

1. For gases, where Pr = (v/α) is of the order of 1, we see that the momentum and heat dissipate almost at the same rate i.e. thicknesses of the hydrodynamic and thermal boundary layers are of the same order;
2. For liquid metals since Pr << 1, it means that heat diffuses at a much higher rate than the momentum for liquid metals i.e. the thermal boundary layer is much thicker than hydrodynamic boundary layer for liquid metals (See Fig 3,a), and,
3. For heavy oils (Pr >> 1), momentum diffuses at a faster rate than heat through the medium and this is evident from Fig. (3,b); thus, the thermal boundary layer is much thinner than hydrodynamic boundary layer.

For laminar conditions, thickness of thermal boundary layer is related to hydrodynamic boundary layer, approximately as follows:

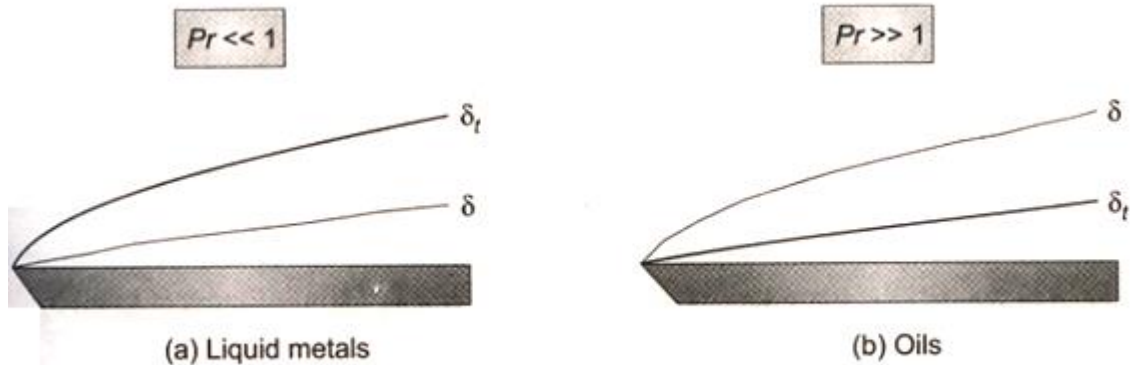$$\frac{\delta_t}{\delta} = \frac{1}{pr^{0.33}}$$

(a) Liquid metals

(b) Oils

Fig. 3 Thermal and velocity boundary layers over a flat plate for liquid metals and oils

6.4 Differential Equations for the Boundary Layer

In convection studies, since there is a fluid flow, we are interested in the shear stress and the friction coefficient; to determine these we need the velocity gradient at the surface. Similarly, to determine the convection coefficient; we need the temperature gradient at the surface. To determine the velocity gradient at the surface, we apply the equation of conservation of momentum (in conjunction with the equation of conservation of mass) to a differential volume element in the boundary layer. And, to determine the temperature gradient at the surface, we apply the equation of conservation of energy to a differential volume element in the boundary layer We start with the application of equation for conservation of mass:

1. Conservation of Mass-The Continuity Equation for The Boundary Layer

Consider a differential control volume, of section $(dx.dy)$ and unit depth, within the boundary layer, as shown in fig. 4
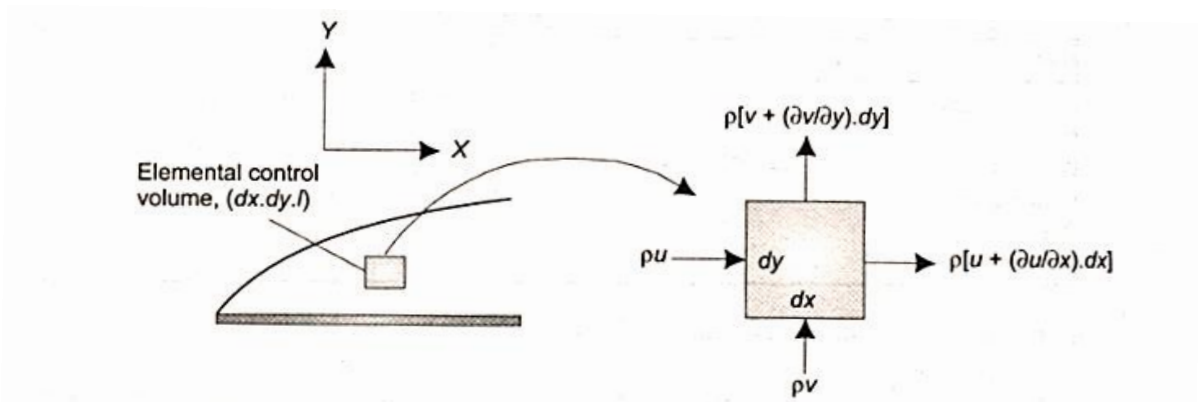

FIGURE 4   Elemental control volume in the boundary layer over a flat plate for conservation of mass

$$\frac{du}{dx} + \frac{dv}{dy} = 0 \dots \dots \dots \dots . a$$

Where: u and v are the velocity components in the X and Y-directions.

Eq. a is known as 'continuity equation' for two-dimensional, steady flow of an incompressible fluid.

2. Conservation of Momentum Equation for The Boundary Layer

This is obtained by the application of Newton's second law of motion to the differential element, which states that the net force on the element in the X-direction is equal to the net momentum from the control volume in the X-direction. Fig. 5 shows the momentum fluxes and forces acting on the differential control volume.

$$u.\left(\frac{du}{dx}\right) + v.\left(\frac{dv}{dy}\right) = v.\left(\frac{d^2u}{dy^2}\right) \dots \dots \dots \dots b$$

Eq. b is known as 'conservation of momentum equation' for two-dimensional, steady flow of an incompressible fluid, with negligible pressure variation in the X-direction
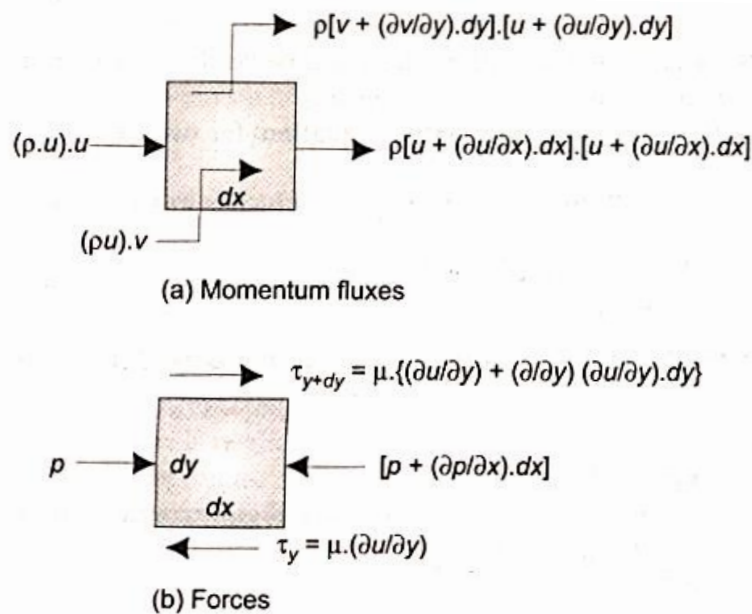

(a) Momentum fluxes

(b) Forces

Figure 5 Conservation of momentum in a two-dimensional, incompressible boundary layer

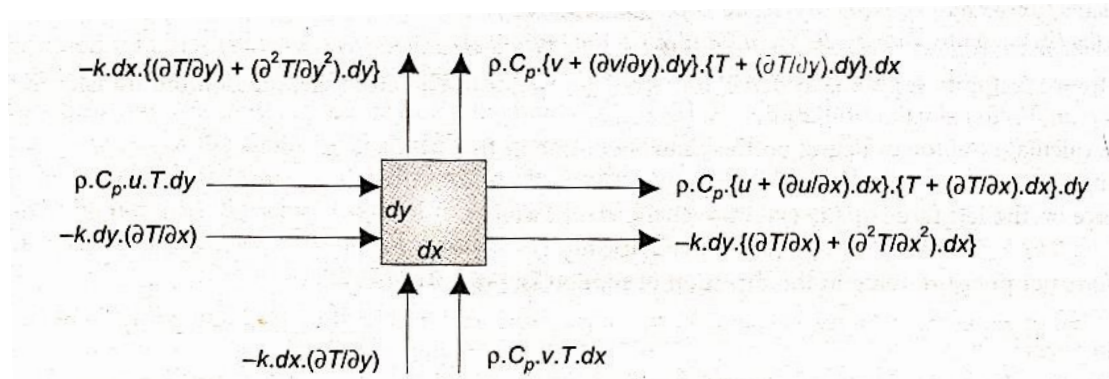## 3. Conservation of Energy Equation for The Boundary Layer

$$u.\left(\frac{dT}{dx}\right) + v.\left(\frac{dT}{dy}\right) = \alpha.\left(\frac{d^2u}{dy^2}\right) \dots \dots \dots \dots c$$

This is the energy equation for a two-dimensional, steady incompressible flow, when the viscous dissipation neglected, i.e. for very low velocities of flow.

Observe the similarity between Eq. b for momentum balance and the Eq. c for energy balance. In Eq. b, $v = \mu/\rho$ = kinematic viscosity, also known as momentum diffusivity. In Eq. c, $\alpha = k/\rho C_p$ is the diffusivity of heat. Their ratio is known as Prandtl number and is equal to:

$$pr = v/\alpha = (\mu/\rho)/(k/\rho C_p) = \mu. C_p/k$$

If $v = \alpha$, then Pr $= 1$ and the momentum and energy equations are identical; thus, Prandtl number controls the relation between the velocity and temperature distributions.

FIGURE 6   Conservation of energy in a two-dimensional, incompressible boundary layer

6.4.1 Exact Solutions of Boundary Layer Equations

Recollect that the equations of continuity, momentum and energy for the boundary layer on a flat plate are given, respectively, by:

$$\frac{du}{dx} + \frac{dv}{dy} = 0 \dots\dots\dots\dots. a$$

$$u.\left(\frac{du}{dx}\right) + v.\left(\frac{dv}{dy}\right) = v.\left(\frac{d^2u}{dy^2}\right) \dots\dots\dots\dots. b$$

$$u.\left(\frac{dT}{dx}\right) + v.\left(\frac{dT}{dy}\right) = \alpha.\left(\frac{d^2u}{dy^2}\right) \dots\dots\dots\dots. c$$

Now, solving the momentum equation in conjunction with the continuity equation gives the velocity distribution, boundary layer thickness and shear stress (or friction force) at the surface. Exact mathematical solution is rather complex; its outline is given below, <u>for the laminar boundary layer ($Re < 5 * 10^5$):</u>

1-

$$\delta = \frac{5.x}{\sqrt{Re_x}}$$

Where: δ is boundary layer thickness.

       x is the distance from the leading edge.

      $Re_x = \frac{\rho U x}{\mu}$ local value of Reynolds number.

2-

$$\tau = 0.332.\mu.\frac{U}{x}.\sqrt{Re_x}$$

τ: is the wall shear stress

$$C_{fx} = \frac{\tau}{\left(\frac{\rho U^2}{2}\right)} = \frac{0.664}{\sqrt{Re_x}}$$

C_{fx} is the local value of the friction coefficient (or the drag coefficient)

$$C_{fa} = 2.C_{fx} = \frac{1.328}{\sqrt{Re_L}}$$

C_{fa} is the average friction coefficient (or the drag coefficient)

3- The relationship between the thickness of hydrodynamic and thermal boundary layer:

$$\frac{\delta_t}{\delta} = \frac{1}{pr^{0.33}}$$

$$\delta_t = \frac{5.x}{\sqrt{Re_x}.Pr^{0.333}}$$

4-

$$Nu_x = \frac{h.x}{k} = 0.332.\sqrt{Re_x}.Pr^{0.333}$$

$$Nu_a = 0.664.\sqrt{Re_L}.Pr^{0.333}$$

$$h = 0.332.\frac{k}{x}.\sqrt{Re_x}.Pr^{0.333}$$

$$h_a = 2.h = 0.664.\frac{k}{x}.\sqrt{Re_L}.Pr^{0.333}$$

Where Nu_x and Nu_a is the local and average Nusselt number, h and h_a is the local and average heat transfer coefficient.

In the above equations, properties of the fluid are evaluated at the mean temperature between the free temperature and the plate surface temperature i.e. at the 'film temperature' given by:

$$T_f = \frac{T_s + T_a}{2}$$

6.5 Relation between the fluid friction and heat transfer coefficient in laminar flow for a flat plate.

Recollect that the average Nusselt number for laminar flow over a flat plate is given by:

$$Nu_a = 0.664.\sqrt{Re_L}.Pr^{0.333}$$

This can be rewritten as:

$$\frac{Nu_a}{Re_L.Pr} = 0.664.Re_L^{-\frac{1}{2}}.Pr^{-\frac{2}{3}}$$

Or:

$$St_a = 0.664.Re_L^{-\frac{1}{2}}.Pr^{-\frac{2}{3}}$$

Where St_a is dimensionless number known as "Stanton number", $St_a = \frac{h_a}{\rho.Cp.U}$

We can write:

$$St_a.Pr^{\frac{2}{3}} = 0.664.Re_L^{-\frac{1}{2}} \dots\dots\dots\dots. a$$

We already know that:

$$C_{fa} = \frac{1.328}{\sqrt{Re_L}}$$

Or:

$$\frac{C_{fa}}{2} = 0.664.Re_L^{-\frac{1}{2}} \dots\dots\dots. b$$

Comparing Eqs. A and b, we can write:

$$St_a . Pr^{\frac{2}{3}} = \frac{C_{fa}}{2}$$

This relation is known as 'Colburn analogy' and it gives a simple relation between the heat transfer coefficient and the friction coefficient.

Note the important significance of this analogy: just by knowing the friction coefficient, one can predict the heat transfer coefficient for that situation; and conducting experiments to determine friction coefficient is many times, practically much easier than conducting experiments to determine heat transfer coefficients.

Dimensionless Form

In a situation of forced convection the significant parameters are $h, \rho, k, \mu, cp, L \ and \ V$. Using Buckingham's Pi-Theorem, we obtain the following three dimensionless numbers:

a) Nusselts number. $.Nu = \frac{h.d}{k}$

b) Reynolds number. $Re = \frac{\rho V L}{\mu}$

c) Prandtl number. $Pr = \frac{\mu c_p}{k}$

EMPIRICAL AND PRACTICAL RELATIONS FOR FORCED-CONVECTION HEAT TRANSFER

it is not always possible to obtain analytical solutions to convection problems, and the individual is forced to resort to experimental methods to obtain design information, as well as to secure the more elusive data that increase the physical understanding of the heat-transfer processes. Results of experimental data are usually expressed in the form of either empirical formulas or graphical charts so that they may be utilized with a maximum of generality

6.6 Empirical And Practical Relations For External Flow

6.6.1 Turbulent Boundary Layer Flow Over A Flat Plate i.e. (Re > 5 * 10^5)

$$Nu = \frac{h.L}{k} = 0.036.Re_L^{0.8}.Pr^{0.33}$$

6.6.2   Flow across Tube Banks

Because many heat-exchanger arrangements involve multiple rows of tubes, the heat transfer characteristics for tube banks are of important practical interest. The heat-transfer characteristics of staggered and in-line tube see fig (7) banks were studied by Zukauskas, and on the basis of a correlation of the results of various investigators, he was able to represent data in the form of the equation below.

$$Nu = C\,Re_{max}^n\,Pr^{0.36}\left(\frac{Pr}{Pr_w}\right)^{0.25}$$

where all properties except Pr_w are evaluated at T_∞ and the values of the constants (C, n) are given in Table (1) for greater than 20 rows of tubes. For less than 20 rows in the direction of flow the correction factor in Table (2) should be applied. This equation is applicable for $0.7<Pr<500$ and $10<Re_{d,\,max}<10^6$. For gases the Prandtl number ratio has little influence and is dropped. note that the Reynolds number is based on the maximum velocity in the tube bundle. that is, the velocity through the minimum-flow area. This area will depend on the geometric tube arrangement.

Determination of maximum flow velocity
For flows normal to in-line tube banks the maximum flow velocity will occur through the minimum frontal area $(S_n -d)$ see fig (7) presented to the incoming free stream velocity u_∞. Thus:

$u_{max} = u_\infty[S_n/(S_n -d)]$ (in-line arrangement)

For the staggered arrangement the same maximum flow velocity will occur through   the minimum frontal area $(S_n -d)$, or this may not be the case for close spacing tube bank, as when S_p is small. For the staggered case, the flow enters the tube bank through the area S_n-d and then splits into the two areas $[(S_n/2)^2 +S_p^2]^{1/2} -d$. If the sum of these two areas is less than $S_n -d$, then they will represent the minimum flow area and the maximum velocity in the tube bank will be:

$$u_{max} = \frac{u_\infty(S_n/2)}{[(S_n/2)^2 + S_p^2]^{1/2} - d}$$

where, again, u_∞ is the free-stream velocity entering the tube bank.

Pressure drop for flow of gases over a bank of tubes may be calculated with the following equation expressed in Pascal:

$$\Delta p = \frac{2 f G_{max}^2 N}{\rho} \left(\frac{\mu_w}{\mu_b}\right)^{0.14}$$

G_{max} =mass velocity at minimum flow area, kg/m^2.s
ρ=density evaluated at free-stream conditions, kg/m^3
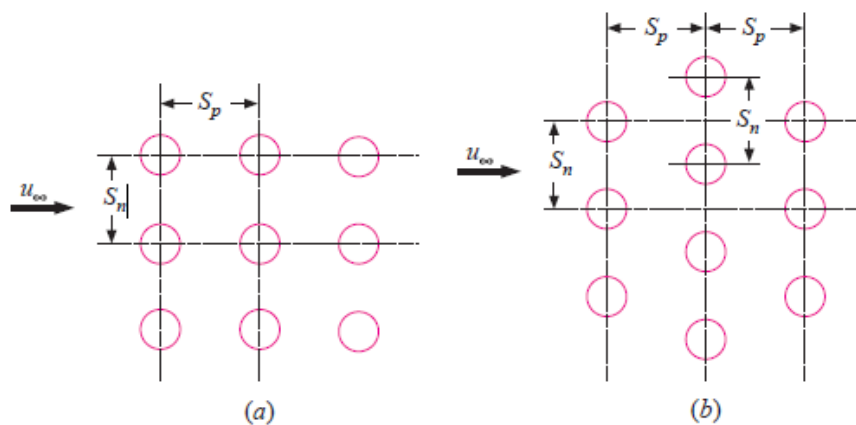N =number of transverse rows
μ_b =average free-stream viscosity, N.s/m^2
The empirical friction factor f is given by Jakob as

$$f = \left[0.25 + \frac{0.118}{[(S_n - d)/d]^{1.08}}\right] Re_{max}^{-0.16} \qquad for\ staggered\ tube\ arrangements$$

$$f = \left[0.044 + \frac{0.08(S_p/d)}{[(S_n - d)/d]^{0.43+1.13d/S_p}}\right] Re_{max}^{-0.15} \qquad for\ in-line\ arrangements.$$

Figure 7 | (a) in-line tube   (b) staggered tube rows.

(a)   (b)

Table 1 | Constants for Zukauskas correlation [Equation   ] for heat transfer in tube banks of 20 rows or more.

Geometry	$Re_{d,max}$	C	n
In-line	10–100	0.8	0.4
	100–10^3	Treat as individual tubes	
	10^3 – 2 × 10^5	0.27	0.63
	> 2 × 10^5	0.21	0.84
Staggered	10–100	0.9	0.4
	100–10^3	Treat as individual tubes	
	10^3 – 2 × 10^5	$0.35 \left(\dfrac{S_n}{S_L}\right)^{0.2}$ for $\dfrac{S_n}{S_L} < 2$	0.60
	10^3 – 2 × 10^5	0.40 for $\dfrac{S_n}{S_L} > 2$	0.60
	> 2 × 10^5	0.022	0.84

Table 2 | Ratio of h for N rows deep to that for 20 rows deep

N	2	3	4	5	6	8	10	16	20
Staggered	0.77	0.84	0.89	0.92	0.94	0.97	0.98	0.99	1.0
In-line	0.70	0.80	0.90	0.92	0.94	0.97	0.98	0.99	1.0

*The reader should keep in mind that these relations correlate experimental data with an uncertainty of about (-,+)25 percent.

6.7 Empirical And Practical Relations For Internal Flow

6.7.1 Flow Inside Tubes

Circular tubes are the most commonly used geometry for cooling and heating applications, in industry. Often, tubes of other geometries such as square or rectangle are also used. We are interested in heat transfer in such cases; pressure drop occurring during flow is also of interest since it has a direct bearing on the pumping power required to cause the flow.

Observe the major difference between the external flows just studied and the internal flow through pipes: in the external flow, say over a flat plate, there was a free surface of fluid and the boundary layer was free to grow; however, in a pipe flow, the flow is confined within the pipe and the boundary layer growth is limited to grow only up to the centre of the pipe.

Hydrodynamic and Thermal Boundary Layers for Flow in a Tube

consider a fluid entering into a circular pipe, with a uniform velocity U See Fig.(8a). Fluid layer coming in contact with the pipe surface comes to a complete halt and the adjacent layers slow down gradually due to viscosity effects. Since the total mass flow in a section must remain constant, velocity in the central portion increases. As a result a 'velocity boundary layer' develops along the pipe. Thickness of the velocity boundary layer increases along the flow length until the entire pipe is filled up with the boundary layer, as shown. 'Hydrodynamic entry length (L_h)' is the distance from the entry point to the point where the boundary layer has developed up to the centre. In the region beyond the hydrodynamic entry length, the velocity profile is fully developed and remains unchanged; this is the 'hydrodynamically developed region'.

Similarly, when a fluid at an uniform temperature enters a pipe whose wall is at different temperature, a 'thermal boundary layer' develops along the pipe. Thickness of thermal boundary layer also increases along the flow length till the boundary layer reaches the centre of the pipe. 'Thermal entry length (L_t)' is the distance from the entry to the point where the thermal boundary layer has reached the centre, and is shown in the Fig. (8b). Beyond this point, along the length, we have the 'fully developed flow' i.e. the flow is both hydrodynamically and thermally fully developed.
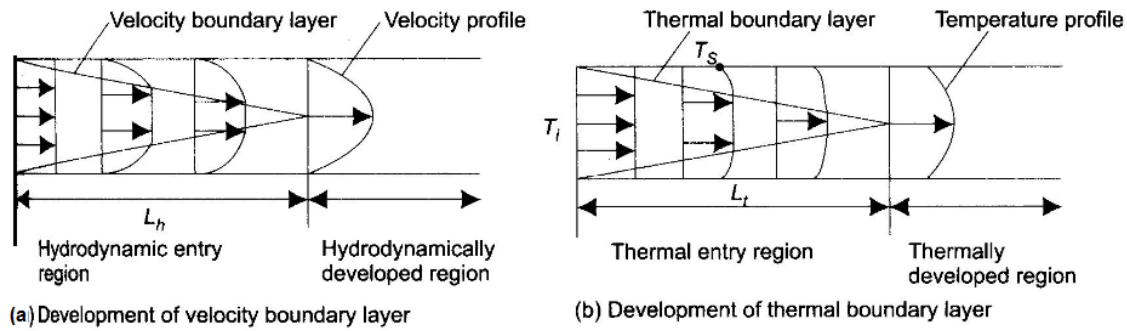
FIGURE 8 Flow inside a pipe

Relative growth of hydrodynamic and thermal boundary layers is controlled by the dimensionless Prandtl number. For gases, Pr =1, and the hydrodynamic and thermal boundary layers essentially coincide; for oils Pr >> 1 and the hydrodynamic boundary layer outgrows the thermal boundary layer, i.e. hydrodynamic entry smaller for oils. For fluids with Pr << 1, such as liquid metals, thermal boundary layer outgrows the hydrodynamic boundary layer and consequently, the thermal entry length is shorter than the hydrodynamic entry.

Reynolds number is the dimensionless number that characterizes the flow inside a tube as laminar or turbulent. Reynolds number is defined as:

$$Re = (U_m . D)/v$$

where U_m is the mean velocity in the pipe, and v is the kinematic viscosity of the fluid. Flow regimes are defined as follows, depending upon the Reynolds number:

$$Re < 2300 \qquad Laminar\ flow$$
$$Re > 2300 \qquad Turbulent\ flow$$

Since the velocity profile remains essentially constant in the hydrodynamically developed region the friction factor and the shear stress remain constant in the hydrodynamically developed region By a similar argument, heat transfer coefficient also remains constant in the thermally developed region.

Correlation for fully developed, steady, laminar flow (i.e. Re < 2300)

    1- Pressure drop

$$\frac{\Delta p}{L} = \frac{f}{D} \frac{\rho \mu_m^2}{2}$$

Where $\Delta p = p1 - p2,\ L = x2 - x1\ and\ f = \frac{64}{Re_D}$

    2-

$$Nu = \frac{h.D}{k} = 1.67 \left[\frac{Re.Pr}{L/D}\right]^{0.333}$$

In the above equation
    a) property values are taken at mean bulk temperature. If the outlet temperature is not specified, iterative working will be required
    b) it is for short pipe, constant wall temperature

3-

$$Nu = 4.36 + \frac{0.036.\left(\frac{D}{L}\right).Re.Pr}{1 + 0.0011.\left[\left(\frac{D}{L}\right).Re.Pr\right]}$$

for short pipe, constant heat flux

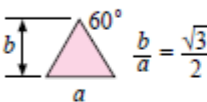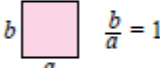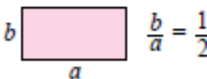for long pipe with constant heat flux, average Nusselt number approaches the value 4.36

4- For non-circular Cross-sections. Nusselts number and friction factor for fully developed laminar flow inside pipes of non-circular cross-sections are given in Table (3). Here, Reynolds number and Nusselts number are based on the hydraulic diameter

$$D_h = \frac{4.A}{P}$$

A is the area of cross-section and P is the wetted perimeter.

Flow through an annulus $D_h = (D_o - D_i)$

Table 3 | Heat transfer and fluid friction for fully developed laminar flow in ducts of various cross sections. Average Nusselt numbers based on hydraulic diameters of cross sections.

Geometry $(L/D_h > 100)$	Nu_H Constant axial wall heat flux	Nu_T Constant axial wall temperature	$f\,Re_{DH}/4$
60° $\frac{b}{a} = \frac{\sqrt{3}}{2}$ (triangle)	3.111	2.47	13.333
$\frac{b}{a} = 1$ (square)	3.608	2.976	14.227
(hexagon)	4.002	3.34	15.054
$\frac{b}{a} = \frac{1}{2}$ (rectangle)	4.123	3.391	15.548
(circle)	4.364	3.657	16.000

Correlation for fully developed, steady, turbulent flow (i.e. Re > 2300)

1- Pressure drop

$$\frac{\Delta p}{L} = \frac{f}{D}\frac{\rho u_m^2}{2}$$

Friction factor f for smooth pipes is given by the following empirical relations:

$$f = 0.316\,Re^{-0.25} \qquad \qquad (\text{for } 2*10^4 < Re < 8 \times 10^4)$$
$$f = 0.184\,Re^{-0.2} \qquad \qquad (\text{for } 10^4 < Re < 10^5)$$
$$f = (0.79\,ln(Re) - 1.64)^{-2} \qquad (for\ 3000 < Re < 5*10^6)$$

Friction factor f for rough pipes:

$$f = \frac{1.325}{\left[ln\left(\frac{\varepsilon}{3.7\,D}\right) + \frac{5.74}{Re^{0.9}}\right]^2}$$

ε is 'roughness height'

2- Design equations

more popularly used design equation for fully developed (L/D > 60), turbulent flow in pipes is the *'Dittus-Boelter equation'*, given below:

$$Nu = 0.023.\,Re^{0.8}.\,Pr^n \qquad (for\ 0.7 < Pr < 160, Re > 10,000)$$

where n = 0.4 for heating and n = 0.3 for cooling of the fluid flowing through the pipe. Here, fluid properties evaluated at the <u>bulk mean temperature</u> of fluid i.e. at $T_b = (T_i + T_e)/2$, where T_i is the temperature of fluid at pipe inlet and T_e is the temperature of fluid at pipe outlet.

If the temperature difference, (T_s - T_b) is significant, then variations in physical properties have to be taken into account, and in such situations correlation of Sieder and Tate   is recommended:

$$Nu = 0.027Re^{0.8}\,Pr^{0.33}\left(\frac{\mu_b}{\mu_w}\right)^{0.14} \qquad (for\ 0.7 < Pr < 10000, 6000 < Re < 10^7)$$

3- For flow through an annulus area use $D_h = (D_o - D_i)$ instead of D

And $f_{annulus} = 0.085\,Re^{-0.25}$ (Re based on hydraulic D_h diameter)

6.7.3   Helically Coiled Tubes

Coiled tubes are used to enhance the heat transfer coefficient and also to accommodate a larger heat exchange surface in a given volume. Heat transfer in a coiled tube is more compared to that in a straight tube due to the contribution of secondary vortices formed as a result of centrifugal forces.

Here, we define a new dimensionless number, called 'Dean number, D_n' as follows:

$$D_n = Re\left(\frac{D}{d_c}\right)^{0.5}$$

Where D is the diameter of the tube and d_c is the diameter of the coil.

For <u>laminar flow</u>, following equations are recommended, depending upon the Dean number:

$$Nu = 1.7(D_n^2.\,Pr)^{\frac{1}{6}} \qquad when\ (D_n < 20)$$

$$Nu = 0.9(Re^2.\,Pr)^{\frac{1}{6}} \qquad When\ (20 < D_n < 100)$$

$$Nu = 0.7(Re^{0.43}.\,Pr^{1/6})\left(\frac{D}{d_c}\right)^{0.07} \qquad When\ (100 < D_n < 830)$$

All the above three Eqs are valid for 10 < Pr < 600.

- Also, for coiled tubes, there is not much difference in values of average Nusselt numbers whether the surface temperature is kept constant or the surface heat flux is maintained constant.
- For simplicity take the critical Reynolds number for a curved pipe as the same that for a straight pipe

For <u>turbulent flow</u>, in forced convection in helically coiled tubes, Hausen has proposed the following correlation:

$$\frac{Nu_{helical}}{Nu_{straight}} = 1 + \left(\frac{21}{Re^{0.14}}\right)\left(\frac{D}{d_c}\right)$$

CH. 6 APPENDIX
Basic Equation For Forced Convection

Geometry, details	Correlation	Restrictions
Flat plate, laminar flow:		$Re < 5 * 10^5$
Hydrodynamic boundary layer thickness	$$\delta = \frac{5.x}{\sqrt{Re_x}}$$	$Re < 5 * 10^5$
local friction coefficient	$$C_{fx} = \frac{\tau}{\left(\frac{\rho U^2}{2}\right)} = \frac{0.664}{\sqrt{Re_x}}$$	$Re < 5 * 10^5$
Average friction coefficient	$$C_{fa} = 2.C_{fx} = \frac{1.328}{\sqrt{Re_L}}$$	$Re < 5 * 10^5$
Local Nusselt number	$Nu_x = 0.332.\sqrt{Re_x}.Pr^{0.333}$	$Re < 5 * 10^5$
Average Nusselt number	$Nu_a = 0.664.\sqrt{Re_L}.Pr^{0.333}$	$Re < 5 * 10^5$
Flat plate, turbulent flow:		$Re > 5 * 10^5$
Nusselt number	$Nu = 0.036.Re_L^{0.8}.Pr^{0.33}$	$Re > 5 * 10^5$
Flow across Tube Banks:		
Nusselt number	$$Nu = C\,Re_{max}^n\,Pr^{0.36}\left(\frac{Pr}{Pr_w}\right)^{0.25}$$	1- (C, n) are given in Table(1) 2-For less than 20 rows the correction factor in Table (2) should be applied
Maximum velocity	$u_{\max} = u_\infty[S_n/(S_n - d)]$ *for* in-line arrangement	
	$$u_{max} = \frac{u_\infty(S_n/2)}{[(S_n/2)^2 + S_p^2]^{1/2} - d}\,for\,staggered$$	
Pressure drop	$$\Delta p = \frac{2fG_{max}^2 N}{\rho}\left(\frac{\mu_w}{\mu_b}\right)^{0.14}$$ $G_{max} = \rho . u_{max}$ N =number of transverse rows	
The friction factor	$$f = \left[0.25 + \frac{0.118}{[(S_n - d)/d]^{1.08}}\right]Re_{max}^{-0.16}$$ *for* in-line arrangement	
	$$f = \left[0.044 + \frac{0.08(S_p/d)}{[(S_n - d)/d]^{0.43+1.13d/S_p}}\right]Re_{max}^{-0.15}$$ *for staggered*	
Flow Inside Tubes, laminar flow:		
Nusselt number	$$Nu = \frac{h.D}{k} = 1.67\left[\frac{Re.Pr}{L/D}\right]^{0.333}$$	Re < 2300
Pressure drop	$$\frac{\Delta p}{L} = \frac{f}{D}\frac{\rho \mu_m^2}{2}$$	
The friction factor	$$f = \frac{64}{Re_D}$$	Re < 2300
Flow Inside Tubes, turbulent flow:		Re > 2300
Nusselt number	$Nu = 0.023.Re^{0.8}.Pr^n$ n = 0.4 for heating	Re > 2300

	n = 0.3 for cooling of the fluid	
Pressure drop	$$\frac{\Delta p}{L} = \frac{f}{D}\frac{\rho \mu_m^2}{2}$$	
The friction factor	$$f = 0.316\,Re^{-0.25}$$ $$f = 0.184\,Re^{-0.2}$$ $$f = (0.79\,ln(Re) - 1.64)^{-2}$$	(for $2*10^4 < Re < 8$ x 10^4) (for $10^4 < Re < 10^5$ ($3000 < Re < 5*10^6$)
Friction factor for rough pipes	$$f = \frac{1.325}{\left[ln\left(\frac{\varepsilon}{3.7\,D}\right) + \frac{5.74}{Re^{0.9}}\right]^2}$$ ε is 'roughness height'	
Friction factor for flow through an annulus area	$$f_{annulus} = 0.085\,Re^{-0.25}$$ (Re based on hydraulic D_h diameter)	
Helically Coiled Tubes, laminar flow:		Re < 2300
Nusselt number	$$Nu = 1.7(D_n^2.Pr)^{\frac{1}{6}}$$ $$Nu = 0.9(Re^2.Pr)^{\frac{1}{6}}$$ $$Nu = 0.7(Re^{0.43}.Pr^{1/6})\left(\frac{D}{d_c}\right)^{0.07}$$ D is the diameter of the tube. d_c is the diameter of the coil.	$(D_n < 20)$ $(20 < D_n < 100)$ $(100 < D_n < 830)$ Where $D_n = Re\left(\frac{D}{d_c}\right)^{0.5}$
Helically Coiled Tubes, turbulent flow:		Re > 2300
Nusselt number	$$\frac{Nu_{helical}}{Nu_{straight}} = 1 + \left(\frac{21}{Re^{0.14}}\right)\left(\frac{D}{d_c}\right)$$	Re > 2300

Table 1 | Constants for Zukauskas correlation [Equation] for heat transfer in tube banks of 20 rows or more.

Geometry	$Re_{d,max}$	C	n
In-line	10–100	0.8	0.4
	100–10^3	Treat as individual tubes	
	$10^3 - 2 \times 10^5$	0.27	0.63
	$> 2 \times 10^5$	0.21	0.84
Staggered	10–100	0.9	0.4
	100–10^3	Treat as individual tubes	
	$10^3 - 2 \times 10^5$	$0.35\left(\frac{S_n}{S_L}\right)^{0.2}$ for $\frac{S_n}{S_L} < 2$	0.60
	$10^3 - 2 \times 10^5$	0.40 for $\frac{S_n}{S_L} > 2$	0.60
	$> 2 \times 10^5$	0.022	0.84

Table 2 | Ratio of h for N rows deep to that for 20 rows deep

N	2	3	4	5	6	8	10	16	20
Staggered	0.77	0.84	0.89	0.92	0.94	0.97	0.98	0.99	1.0
In-line	0.70	0.80	0.90	0.92	0.94	0.97	0.98	0.99	1.0

CHAPTER SEVEN
NATURAL CONVECTION;
Summery and Empirical Relations

Introduction

In the previous chapter, we studied heat transfer by forced convection, wherein fluid movement was caused by an external agency such as a pump or fan. In this chapter, we shall study about heat transfer in 'Natural or free convection'; here, fluid movement is caused because of density differences in the fluid due to temperature differences. Under the influence of gravity, density differences cause a 'buoyancy force' which in turn, causes the fluid circulation by 'convection currents'.

Physical Mechanism of Natural Convection

Consider the familiar example of a heated vertical plate kept hanging in calm air. Let the temperature of the heated surface be Ts and that of the surrounding air, Ta. A layer of air in the immediate vicinity of the plate will get heated by conduction; density of this heated air layer decreases. As a result, the heated layer rises up and the cold air from the surroundings moves in to take its place. This layer, in turn, gets heated up, moves up and is again replaced by cooler air etc. Thus, convection currents are set up causing the heat to be carried away from the hot surface. This situation is shown in Fig.

During the temperature induced flow, a boundary layer is set up along the length of the plate as shown. With the x-axis taken along the vertical length of plate, the velocity and temperature profiles are shown in the Fig. As far as the velocity profile is concerned, at the plate surface, the fluid velocity is zero due to 'no slip' condition; then, the velocity increases to a maximum value and then, drops to zero at the outer edge of the boundary layer since the surrounding air is assumed to be stationary. Note the difference in this velocity profile as compared to that in the case of forced convection. The boundary layer is laminar for some distance along the length, and then depending on the fluid properties and the driving temperature difference between the wall and the ambient, the boundary layer becomes turbulent.
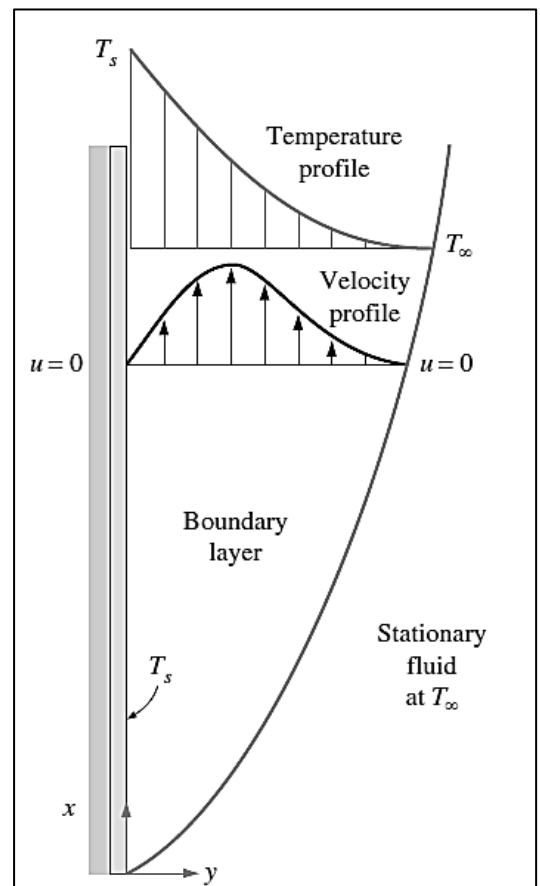
Dimensionless numbers of natural convection

1- Gr = Grashoff number: plays the same role in natural convection as that of Reynolds number in forced convection. Physical significance of Grashoff number is that it represents the ratio of buoyancy force to the viscous force acting on the fluid, i.e.

$$Gr = \frac{\text{buoyancy force}}{\text{viscous force}} = \frac{g.\Delta\rho.V}{\rho.v^2} = \frac{g.\beta.\Delta T.V}{v^2}$$

$$Gr = \frac{g.\beta.(T_s - T_a).L_c^3}{v^2}$$

Where:

g = acceleration due to gravity, m/s^2

β = coefficient of volume expansion, 1/K (β = l/T for ideal gases only, T in Kelvin)

Ts = temperature of the surface, deg. C

Ta = temperature of the fluid at sufficient distance from the surface, deg. C

Lc = characteristic length of the geometry, m

v = kinematic viscosity of fluid, m^2/s

2- Prandtl number (Pr) = $\dfrac{\mu\, c_p}{k}$

3- Rayleigh number, (Ra): Is the Product of Grashoff number and Prandtl number, i.e. $Ra = (Gr.Pr)$. It is the criterion to determine if the flow is laminar or turbulent, in natural convection. For example, in the case of heat transfer by natural convection for vertical plates, for Ra > about 10^9, the flow is turbulent and for Ra < 10^9, the flow is laminar.

4- Nusselt number (Nu) = C.Ra^m

Where:

Ra = (Gr.Pr) = Rayleigh number and 'C and 'm' are constants determined from experiments, (depending on the convection case).

By determining Nu, we determine h, the heat transfer coefficient in natural convection. $(Nu = \dfrac{h\, Lc}{k})$

Then, the heat transfer rate for natural convection is given bv Newton's law of cooling, i.e.
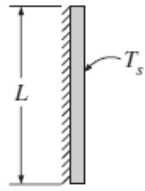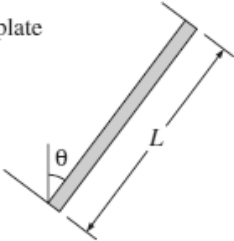
$$Q_{conv} = h.A.(T_s - T_a)$$
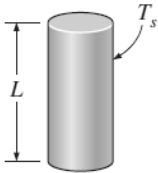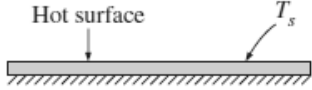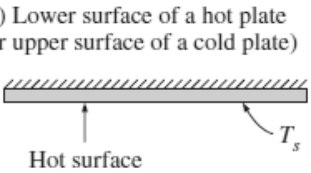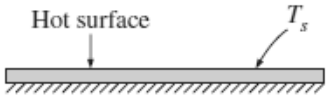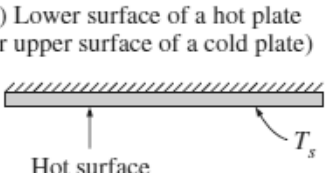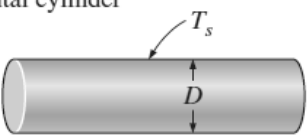
Empirical Relations for Natural Convection over Surfaces and Enclosures

In solving natural convection heat transfer problems, we rely more on empirical relation than on analytical relations. This is due to the fact that analytical relations are rather difficult to obtain also; analytical relations generally give lower values of heat transfer coefficients as compared to empirical relations. Also, because of the very low velocities involved in natural convection, it becomes difficult to take into account all factors in the analytical methods.

(a) Cold vertical surface
(b) Hot surface facing up
(c) Cold surface facing up
(d) Hot surface facing down
(e) Cold surface facing down

Free convection flow patterns
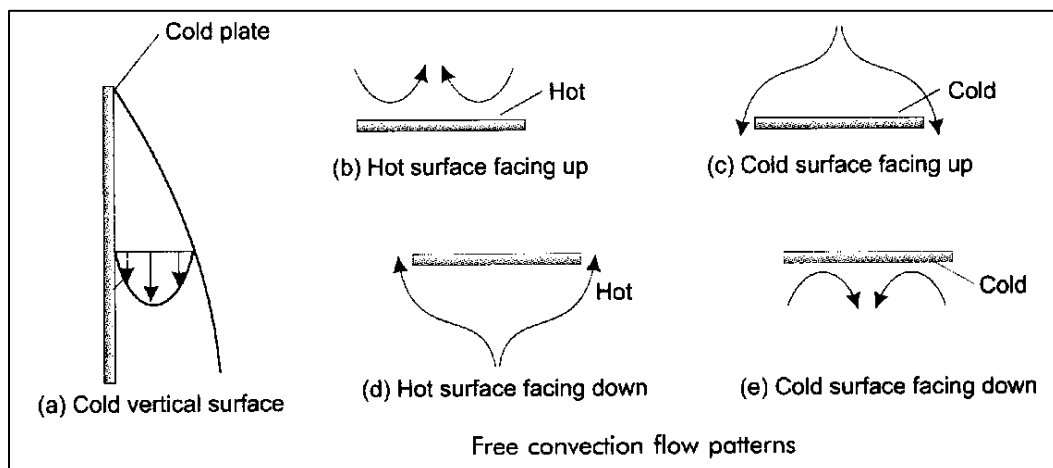
We shall present below empirical relations for natural convection from several types of surfaces and enclosures of practical importance.

	Geometry	Lc	Condition	Range of Ra	Nu
1-	Vertical plate T_s, L	L	Constant (T)	$10^4 - 10^9$ $10^9 - 10^{13}$	$Nu = 0.59\,Ra^{1/4}$ $Nu = 0.13\,Ra^{1/3}$
2-	Inclined plate L, θ	L	Constant (T)	Use vertical plate equations. Replace *(g)* by $(g\cos\theta)$ for $Ra > 10^9$
3-	Vertical cylinder T_s, L	L	Constant (T)	A vertical cylinder can be treated as a vertical plate, if the cylinder diameter is large. For simplicity we will consider all the problem as large diamere
4-	Horiontal plate (Surface area A and perimeter p) (a) Upper surface of a hot plate (or lower surface of a cold plate) Hot surface T_s (b) Lower surface of a hot plate (or upper surface of a cold plate) T_s Hot surface	$L_c = A/P$ Where: A= surface area P = perimeter	Constant (T)	$10^4 - 10^7$ $10^7 - 10^{11}$ $10^5 - 10^{11}$	$Nu = 0.54\,Ra^{1/4}$ $Nu = 0.15\,Ra^{1/3}$ $Nu = 0.59\,Ra^{1/4}$
5-	Horiontal plate (Surface area A and perimeter p) (a) Upper surface of a hot plate (or lower surface of a cold plate) Hot surface T_s (b) Lower surface of a hot plate (or upper surface of a cold plate) T_s Hot surface	$L_c = A/P$ Where: A= surface area P = perimeter	Constant (Q)	$< 2 \times 10^8$ $2 \times 10^8 - 10^{11}$ $10^6 - 10^{11}$	$Nu = 0.13\,Ra^{1/3}$ $Nu = 0.16\,Ra^{1/3}$ $Nu = 0.58\,Ra^{0.2}$
6-	Horizontal cylinder T_s, D	D	Constant (T)	$10^{-5} - 10^{12}$	$Nu = \left[0.6 + \dfrac{0.387\,Ra^{\frac{1}{6}}}{\left[1 + \left(\frac{0.559}{Pr}\right)^{\frac{9}{16}}\right]^{0.3}} \right]^2$

Example: Consider a flat-plate solar collector placed horizontally on the flat roof of a house. The collector is 1.5 m wide and 6 m long, and the average temperature of the exposed surface of the collector is 42°C. Determine the rate of heat loss from the collector by natural convection during a calm day when the ambient air temperature is 8°C.

Solution:

A flat-plate solar collector placed horizontally on the flat roof of a house is exposed to the calm ambient air. The rate of heat loss from the collector by natural convection and radiation are to be determined.

Properties The properties of air at 1 atm and the film temperature of $(T_s + T_\infty)/2 = (42+8)/2 = 25°C$ are

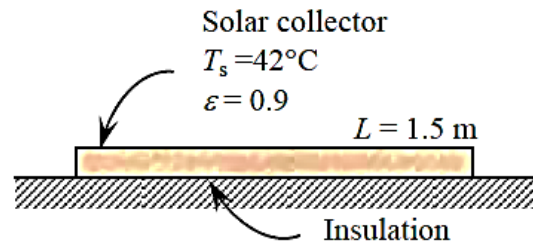$$k = 0.02551 \text{ W/m.°C}$$

$$v = 1.562 \times 10^{-5} \text{ m}^2/\text{s}$$

$$\text{Pr} = 0.7296$$

$$\beta = \frac{1}{T_f} = \frac{1}{(25+273)\text{K}} = 0.003356 \text{ K}^{-1}$$

Air
$T_\infty = 8°C$

Solar collector
$T_s = 42°C$
$\varepsilon = 0.9$
$L = 1.5 \text{ m}$

Insulation

Analysis The characteristic length in this case is determined from

$$L_c = \frac{A_s}{p} = \frac{(1.5 \text{ m})(6 \text{ m})}{2(1.5 \text{ m} + 6 \text{ m})} = 0.6 \text{ m}$$

Then,

$$Ra = \frac{g\beta(T_\infty - T_s)L_c^3}{v^2} \text{Pr} = \frac{(9.81 \text{ m/s}^2)(0.003356 \text{ K}^{-1})(42-8 \text{ K})(0.6 \text{ m})^3}{(1.562 \times 10^{-5} \text{ m}^2/\text{s})^2}(0.7296) = 7.230 \times 10^8$$

$$Nu = 0.15 Ra^{1/3} = 0.15(7.230 \times 10^8)^{1/3} = 134.6$$

$$h = \frac{k}{L_c} Nu = \frac{0.02551 \text{ W/m.°C}}{0.6 \text{ m}}(134.6) = 5.72 \text{ W/m}^2.°C$$

$$A_s = (1.5 \text{ m})(6 \text{ m}) = 9 \text{ m}^2$$

and

$$\dot{Q}_{conv} = hA_s(T_s - T_\infty) = (5.72 \text{ W/m}^2.°C)(9 \text{ m}^2)(42-8)°C = \mathbf{1750 \text{ W}}$$

Example:

A cold cylinder is placed horizontally in hot air. The rates of heat transfer from the stack with and without wind cases are to be determined.
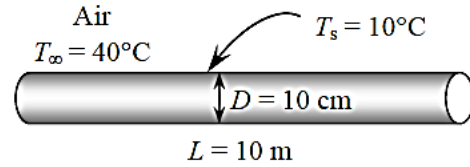
Properties The properties of air at 1 atm and the film temperature of $(T_s+T_\infty)/2 = (40+10)/2 = 25°C$ are

$k = 0.02551 \text{ W/m.°C}$

$v = 1.562 \times 10^{-5} \text{ m}^2/\text{s}$

$\text{Pr} = 0.7296$

$\beta = \dfrac{1}{T_f} = \dfrac{1}{(25+273)\text{K}} = 0.003356 \text{ K}^{-1}$

Air
$T_\infty = 40°C$ $T_s = 10°C$

$D = 10 \text{ cm}$

$L = 10 \text{ m}$

Analysis (*a*) When the stack is exposed to 10 m/s winds, the heat transfer will be by forced convection. We have flow of air over a cylinder and the heat transfer rate is determined as follows:

$\text{Re} = \dfrac{VD}{v} = \dfrac{(10 \text{ m/s})(0.1 \text{ m})}{1.562 \times 10^{-5} \text{ m}^2/\text{s}} = 64,020$

$\text{Nu} = 0.027 \text{Re}^{0.805} \text{Pr}^{1/3} = 0.027(64,020)^{0.805}(0.7296)^{1/3} = 179.8$

$h = \dfrac{k}{D} \text{Nu} = \dfrac{0.02551 \text{ W/m.°C}}{0.1 \text{ m}}(179.8) = 45.87 \text{ W/m}^2.°C$

$\dot{Q}_{\text{forced conv}} = hA(T_\infty - T_s) = (45.87 \text{ W/m}^2.°C)(\pi \times 0.1 \times 10 \text{ m}^2)(40-10)°C = \mathbf{4323 \text{ W}}$

(b) Without wind the heat transfer will be by natural convection. The characteristic length in this case is the outer diameter of the cylinder, $L_c = D = 0.1 \text{ m}$. Then,

$Ra = \dfrac{g\beta(T_\infty - T_s)D^3}{v^2} \text{Pr} = \dfrac{(9.81 \text{ m/s}^2)(0.003356 \text{ K}^{-1})(40-10 \text{ K})(0.1 \text{ m})^3}{(1.562 \times 10^{-5} \text{ m}^2/\text{s})^2}(0.7296) = 2.953 \times 10^6$

$Nu = \left\{ 0.6 + \dfrac{0.387 Ra^{1/6}}{\left[1 + (0.559/\text{Pr})^{9/16}\right]^{8/27}} \right\}^2 = \left\{ 0.6 + \dfrac{0.387(2.953 \times 10^6)^{1/6}}{\left[1 + (0.559/0.7296)^{9/16}\right]^{8/27}} \right\}^2 = 19.86$

$h = \dfrac{k}{D} Nu = \dfrac{0.02551 \text{ W/m.°C}}{0.1 \text{ m}}(19.86) = 5.066 \text{ W/m}^2.°C$

$\dot{Q}_{\text{nat. conv}} = hA(T_\infty - T_s) = (5.066 \text{ W/m}^2.°C)(\pi \times 0.1 \times 10 \text{ m}^2)(40-10)°C = \mathbf{477 \text{ W}}$

- CHAPTER EIGHT-

BOILING AND CONDENSATION

8.1 **Introduction**

8.2 **Condensation heat transfer**

8.2.1 **Film condensation and flow regimes**

8.2.2 **Nusselt's theory for laminar film condensation on vertical plates**

8.2.3 **Film condensation on inclined plates, vertical tubes and horizontal tubes**

8.3 **Boiling and evaporation**

8.3.1 **Boiling modes**

8.3.1 **Boiling regimes and boiling curve**

8.3.2 **Burnout phenomenon**

8.3.3 **Simplified relations for boiling heat transfer with water**

CHAPTER EIGHT
BOILING AND CONDENSATION

8.1 Introduction
In the previous chapter we studied convection heat transfer, i.e. heat transfer with fluid motion and the fluid involved was homogeneous and in single phase. But, there are many important practical cases which involve heat transfer with a change of phase of the fluid, e.g. boiling where the liquid changes to vapour and condensation where the vapour condenses into a liquid, Boiling and condensation are classified under convection since there is motion of the fluid during heat transfer in these processes.

Some of the applications of boiling and condensation are:
1. Evaporators and condensers of a vapour compression refrigerating system
2. Boilers and condensers of a steam power plant
3. Reboilers and condensers of distillation columns of cryogenic and petrochemical plants
4. Cooling of nuclear reactors and rocket motors
5. Process heating and cooling, etc

Unique features of boiling and condensation are:
1. heat transfer, practically at a constant temperature, because of change of phase
2. latent heat and surface tension come into play in addition to buoyancy driven flow effects, resulting in larger heat transfer rates and heat transfer coefficients compared to the usual free or forced convection
3. high heat transfer rates with small temperature difference.

Dimensionless Parameters in Boiling and Condensation
It is difficult to obtain governing equations for boiling and condensation by applying the usual conservation law. However, dimensional analysis has been successfully applied with the use of Buckingham π theorem. Heat transfer coefficient in either boiling or condensation process may be reasonably assumed to depend on;

- The temperature difference ΔT between the surface temperature T_s and the saturation temperature T_{sat} of the fluid.
- Body force arising out of the density difference between the liquid and vapour phases = $[g \cdot (\rho_l - \rho_v)]$.
- Latent heat h_{fg}.
- Surface tension σ.
- Characteristic length L.
- Thermo-physical properties of the liquid or vapour; ρ, c_p, μ and k.

By applying Buckingham π theorem one can be get the functional relationship between the various dimensionless groups.

8.2 CONDENSATION HEAT TRANSFER

Condensation heat transfer has important practical applications, e.g. in thermal power plants, chemical plant, refrigeration and air-conditioning systems.

Whenever a saturated vapour at a temperature T_{sat} is brought in contact with a surface maintained at temperature T_s such that T_s is less than T_{sat} vapours condense on the surface. So, condensation is the 'reverse' of boiling process.

While condensing, the vapours will release the latent heat of vaporisation.

The vapours may condense on the surface in one of the two modes: 'film-wise condensation' or 'drop-wise condensation'.

In **film-wise condensation**, vapours condense on the surface and drip down forming a continuous liquid film on the surface. Thickness of the condensate film increases as it travels down towards the lower (or trailing) end of the plate. During the condensation process, latent heat of vaporisation is released by the vapours. For further condensation to occur, the released latent heat has to be conducted through this liquid film to the cooled surface at temperature T_s. However, the liquid film offers resistance to the flow of heat and this resistance increases as the thickness of the film grows. Film-wise condensation occurs on surfaces which tend to get 'wetted'.

In **drop-wise condensation**, the vapours condense on the surface on drops, which drip down the surface. A continuous film of liquid is not formed on the surface. Thus, more of the base area at temperature T_s is always exposed to the vapours. Therefore, heat transfer rate is higher (up to ten times) in drop-wise condensation as compared to the value in film-wise condensation. Generally, drop-wise condensation occurs on smooth surfaces which do not get 'wetted'.

While drop-wise condensation would appear to be the preferred mode in practice, it is difficult to maintain this mode since, with time, all surfaces tend to get wetted.

Attempts to achieve drop-wise condensation have been made either by coating the surface with some suitable material or by adding some additives to the vapours; but, commercially, these techniques have not yet become viable.

(a) Film condensation (b) Dropwise condensation
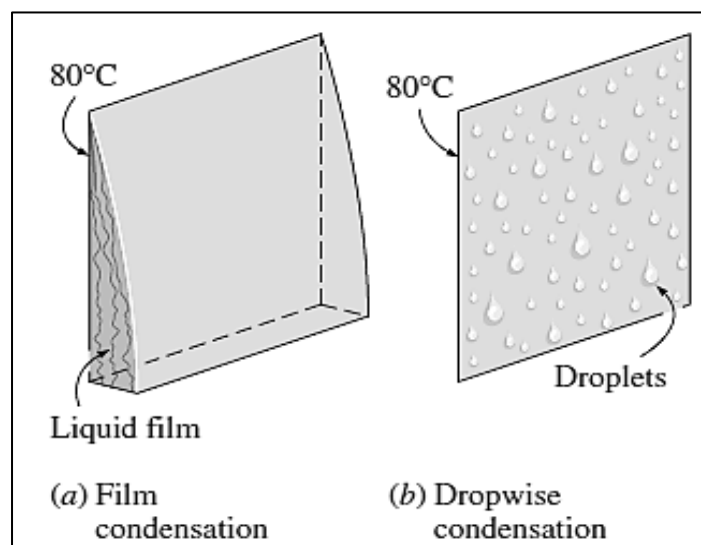
8.2.1Film Condensation and Flow Regimes

Consider film condensation of a vapour at saturation temperature T_{sat} on the surface of a cooled vertical plate, maintained at a temperature T_s ($< T_{sat}$)

Vapour condenses on the top of the plate and flows down as a film. Thickness of the film (δ) is zero at the top of the plate (i.e. at x = 0 in the coordinate system shown in fig.) and increases as we travel down the plate (i.e. as x increases) due to additional condensation of vapour.

Initially, the liquid film flow is laminar; after some distance it will become wavy and later, it may even turn turbulent. These different flow regimes are identified according to a 'film Reynolds number', defined as follows:

$$Re_f = \frac{D_L . \rho_L . V_L}{\mu_L}$$

$$= \frac{4 . A_c . \rho_L . V_L}{P . \mu_L} = \frac{4 . \rho_L . V_L . \delta}{\mu_L} = \frac{4 . m}{P . \mu_L}$$

Where

$D_h = 4 A_c / P = 4.\delta =$ Hydraulic diameter of condensate flow, (m)
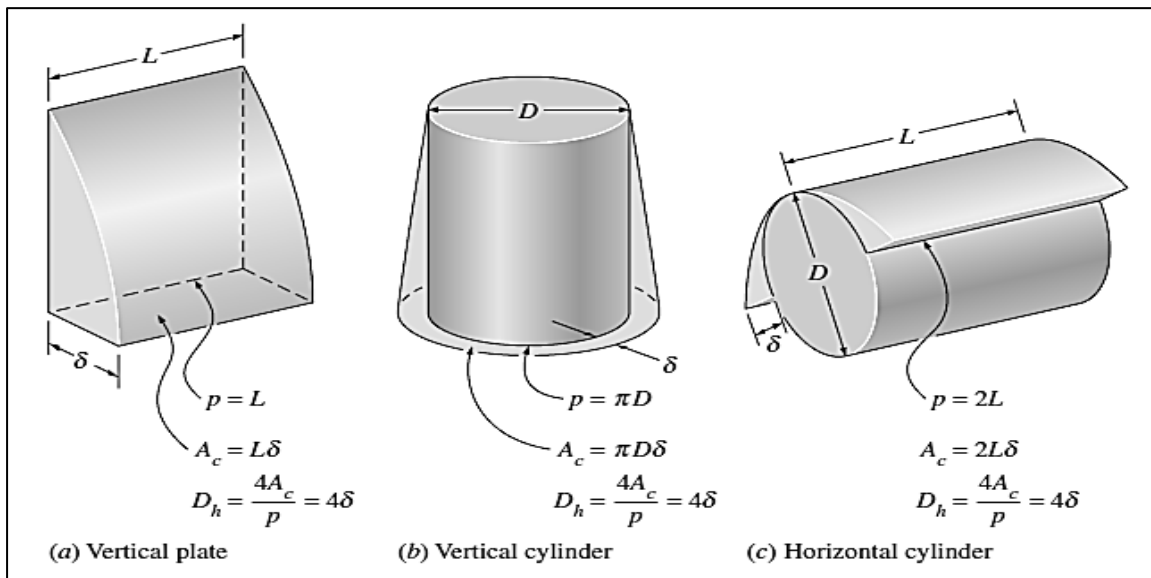
P = wetted perimeter of condensate, (m)

$A_c = P.\delta =$ area of cross section of flow at the lowest part of flow, (m)

$\rho_L =$ density of liquid, kg/m^3

$\mu_L =$ viscosity of liquid, kg/ms

$V_L =$ average velocity of condensate at the lowest part of flow, m/s

$A_c . \rho_L . V_L =$ m = mass flow rate of condensate at the lowest part of flow, kg/s.

(a) Vertical plate   (b) Vertical cylinder   (c) Horizontal cylinder

For the common geometries of a vertical plate, vertical cylinder and a horizontal cylinder, hydraulic diameter D_h is equal to 4 times the thickness of the condensate, δ, at the location where the hydraulic diameter is to be evaluated

If we denote (m/P) by m', we can write for the vertical plate:

$$Re_f = \frac{4 . m'}{\mu_L}$$

Another point to be noted is regarding the latent heat of vaporisation (h_{fg}) released during condensation: Vapour at a temperature of T_{sat} comes in contact with the plate at a temperature of T_s ($<T_{sat}$) and condenses, if the condensed liquid is, further sub-cooled to a temperature

somewhere in between T_s and T_{sat}, thus releasing some more heat. Rohsenow (1956) suggested that this sub-cooling of the liquid can be taken into account by replacing h_{fg} by a 'modified latent heat of vaporisation h'_{fg} defined as:

$$h'_{fg} = h_{fg} + 0.68\, C_{pL}\, (T_{sat} - T_s)$$

Where, C_{pL}, is the specific heat of liquid at the average film temperature.

Similarly, if a superheated vapour at a temperature, T_v, enters a condenser and condenses, the superheated vapour has to be cooled to T_{sat} first, and then condensed at T_{sat} and then sub-cooled to some temperature between T_s and T_{sat} Then, modified latent heat of vaporisation is:

$$h'_{fg} = h_{fg} + 0.68\, C_{pL}\, (T_{sat} - T_s) + C_{pV}\, (T_v - T_{sat})$$

Where, C_{pV} is the specific heat of vapour at the average temperature of $(T_v + T_{sat})/2$.

Then, rate of heat transfer in condensation becomes:

$$Q_{conden} = h \cdot A \cdot (T_{sat} - T_s) = m \cdot h'_{fg}$$

Where, A is the surface area on which condensation occurs.

Then, we can write:

$$Re_f = \frac{4 \cdot Q_{conden}}{P \cdot \mu_L \cdot h'_{fg}} = \frac{4 \cdot h \cdot A \cdot (T_{sat} - T_s)^{\cdot}}{P \cdot \mu_L \cdot h'_{fg}}$$
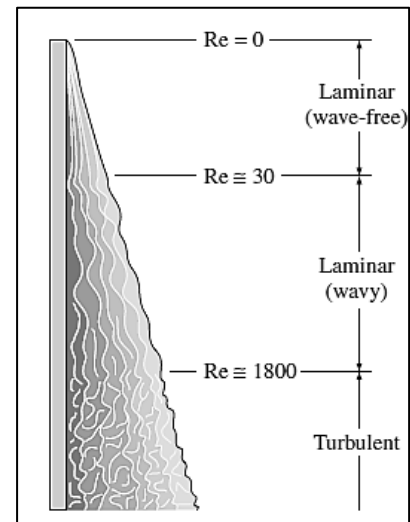
Now, different flow regimes are identified according to the value of Re_f as follows.

$Re_f < 30$ (Liquid film is smooth and wave-free, i.e. fully laminar.)

$450 < Re_f < 1800$ (Liquid film has ripples or waves and the flow is wavy—laminar.)

$Re_f > 1800$ (Liquid film is fully turbulent.)

Heat transfer correlations vary depending upon the flow regime.

8.2.2 Nusselt's Theory for Laminar Film Condensation on Vertical Plates

Nusselt developed his theory tor laminar film condensation on vertical plates analytically in 1916.

Consider a vertical plate maintained at a temperature T_s and exposed to a saturated vapour at a temperature of T_{sat} ($T_s < T_{sat}$). See Fig. 1. Let the height of the plate be L and the breadth b. Coordinate system is chosen such that x-coordinate is to the downward direction, i.e. in the direction of flow of condensate and y-coordinate is towards the right, as shown. Condensation occurs on the plate and the condensate moves down from top to bottom. Thickness of condensate is zero at the top (i.e. at x = 0) and increases in the flow direction, due to additional condensation of vapour. Since the liquid film offers resistance to the flow of heat from the vapour to the cold surface, this also means that resistance to heat transfer is minimum at the top of the plate and the resistance increases as one moves down in the flow direction.

Nusselt made the following simplifying assumptions in his analysis:
1. Flow of liquid film is laminar
2. Heat flow is mainly by conduction through the liquid film
3. Temperature is T_s at the liquid-plate interface and T_{sat} at the liquid-vapour interface and the temperature gradient between them is linear
4. Velocity of vapour is low, i.e. there is no viscous shear force at the liquid-vapour interface
5. Properties of the liquid are constant.

The volume element of condensate on a vertical plate considered in Nusselt's analysis.
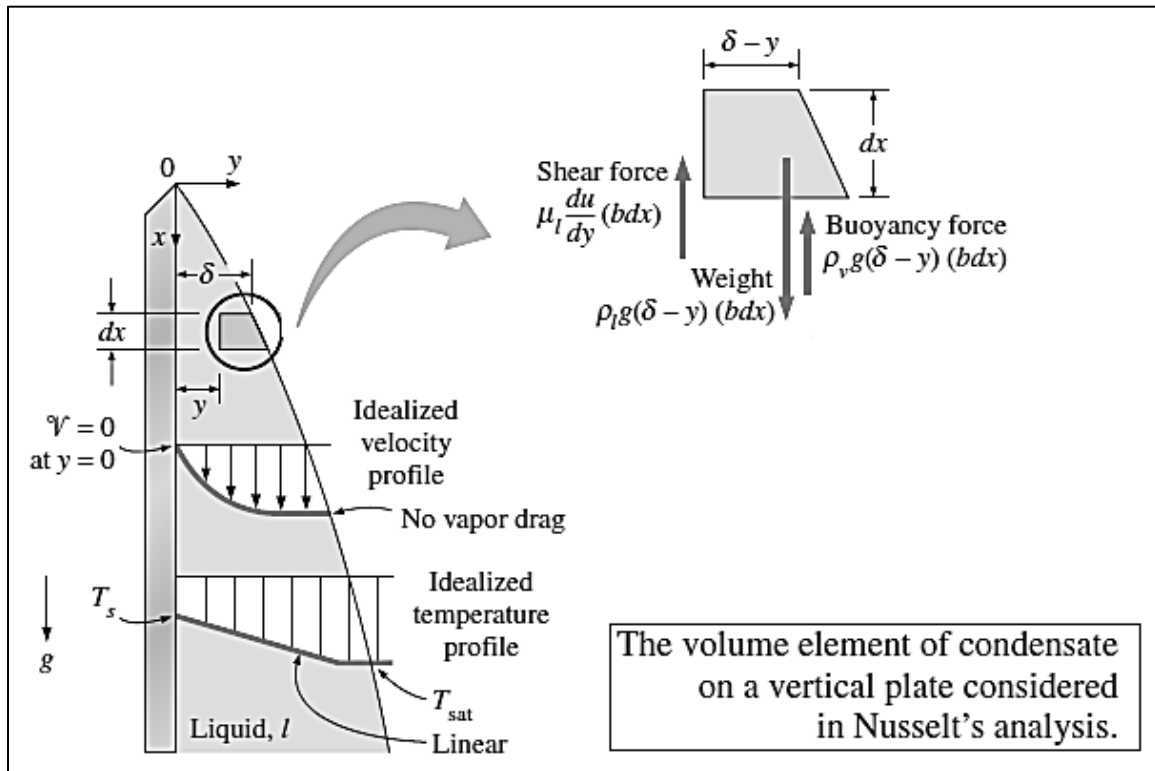
Making the force balance:

$$F_{down} = F_{up}$$

$$Weight = Viscous\ shear\ force\ +\ Buoyancy\ force$$

$$\rho_L \cdot g \cdot (\delta - y) \cdot (b \cdot dx) = \mu_L \cdot \frac{du}{dy} \cdot (b \cdot dx) + \rho_V \cdot g \cdot (\delta - y) \cdot (b \cdot dx)$$

$$\frac{du}{dy} = \frac{g \cdot (\rho_L - \rho_V) \cdot (\delta - y)}{\mu_L}$$

Integrating from y = 0 (i.e. the plate surface) to y = y, and remembering that at y = 0, u = 0, and at y = y, u = u(y), we get:

$$u(y) = \frac{g \cdot (\rho_L - \rho_V) \cdot \left[\delta \cdot y - \frac{y^2}{2}\right]}{\mu_L}$$

The mean flow velocity of the liquid at a section is given by:

$$u_m = \frac{1}{\delta} \cdot \int_0^\delta u\,dy$$

$$u_m = \frac{1}{\delta} \cdot \int_0^\delta \frac{g \cdot (\rho_L - \rho_V) \cdot \left[\delta \cdot y - \frac{y^2}{2}\right]}{\mu_L} dy$$

$$u_m = \frac{g \cdot (\rho_L - \rho_V) \cdot \delta^2}{3 \cdot \mu_L}$$

Mass flow rate:
Mass flow rate of condensate through any x-position is given by:

$$mass\ flow\ rate = density * area * mean\ velocity$$

$$m = \rho_L . (b . \delta) . \frac{g . (\rho_L - \rho_V) . \delta^2}{3 . \mu_L}$$

$$m = \frac{\rho_L (\rho_L - \rho_V) . g . b . \delta^3}{3 . \mu_L} \qquad 2$$

Note that mass flow rate is a function of position (x), since the film thickness δ is function of x. As we proceed from position x to ($x + dx$), film thickness increases from δ to ($\delta + d\delta$), and there is additional mass 'dm' condensed. This additional mass 'dm' condensed between x and (x + dx) is obtained by differentiating Eq. 2

$$dm = \left[\frac{\rho_L (\rho_L - \rho_V) . g . b . \delta^2}{\mu_L} \right]$$

Heat flow rate:

While condensing 'dm' amount of liquid, certain amount of latent heat of vaporisation is released; this is equal to:

$$dQ = dm . h_{fg}$$

$$dQ = h_{fg} . \left[\frac{\rho_L (\rho_L - \rho_V) . g . b . \delta^2}{\mu_L} \right] \qquad 3$$

But, as per the assumption, heat flow through the liquid film is by pure conduction, with linear temperature gradient. Therefore, we can write:

$$dQ = \frac{k_L . (b . dx)}{\delta} . (T_{sat} - T_s) \qquad 4$$

From Eq. 3 and 4:

$$h_{fg} . \left[\frac{\rho_L (\rho_L - \rho_V) . g . b . \delta^2}{\mu_L} \right] = \frac{k_L . (b . dx)}{\delta} . (T_{sat} - T_s)$$

$$\delta^3 = \frac{k_L . \mu_L}{\rho_L (\rho_L - \rho_V) . g . h_{fg}} . (T_{sat} - T_s) . dx$$

Integrating the above equation with the boundary condition that $\delta = 0$ at x = 0, we get:

$$\delta(x) = \left[\frac{4 . k_L . \mu_L . (T_{sat} - T_s) . x}{\rho_L (\rho_L - \rho_V) . g . h_{fg}} \right]^{\frac{1}{4}} \qquad 5$$

Eq. 5 gives the liquid film thickness as a function of position x.

Heat transfer coefficient:

For the heat flow through the liquid film, we have:

$$dQ = \frac{k_L . (b . dx)}{\delta} . (T_{sat} - T_s)$$

Also, by Newton's law of cooling:

$$dQ = h_x . (b . dx) . (T_{sat} - T_s)$$

where, h_x is the local heat transfer coefficient
    From the above two relations, we get:

$$\frac{k_L \cdot (b \cdot dx)}{\delta} \cdot (T_{sat} - T_s) = h_x \cdot (b \cdot dx) \cdot (T_{sat} - T_s)$$

$$h_x = \frac{k_L}{\delta}$$

Substituting the value of δ from Eq. 5 in the above Eq.

$$h_x = \left[\frac{\rho_L \, (\rho_L - \rho_V) \cdot k_L^3 \cdot g \cdot h_{fg}}{4 \cdot \mu_L \cdot L \cdot (T_{sat} - T_s)}\right]^{\frac{1}{4}} \qquad 6$$

Obviously, rate of condensation heat transfer is higher at the upper end as compared to that at the lower end.

Average value of heat transfer coefficient over the entire height of the plate is of interest to calculate the total heat transfer rate. This is obtained by integrating Eq. 6 over the height L:

$$h_{avg} = \frac{1}{L} \cdot \int_0^L h_x dx$$

$$h_{avg} = \frac{4}{3} \cdot h_x$$

Substituting for h_x from Eq. 6 we get:

$$h_{avg} = \frac{4}{3} \cdot \left[\frac{\rho_L \, (\rho_L - \rho_V) \cdot k_L^3 \cdot g \cdot h_{fg}}{4 \cdot \mu_L \cdot L \cdot (T_{sat} - T_s)}\right]^{\frac{1}{4}}$$

$$h_{avg} = 0.943 \cdot \left[\frac{\rho_L \, (\rho_L - \rho_V) \cdot k_L^3 \cdot g \cdot h_{fg}}{\mu_L \cdot L \cdot (T_{sat} - T_s)}\right]^{\frac{1}{4}} \qquad 7$$

Eq. 7 is Nusselt's equation for average heat transfer coefficient for condensation on a vertical plate. It is observed that in practice, experimental value of average heat transfer coefficient is about 20% higher than that given by Nusselt's Eq. 7. So, McAdams suggested to use a coefficient of 1.13 instead of 0.943 in Eq. 7

Nusselts equation under predicts the value of h, basically because:
a) it does not take into account non-linear temperature profile in the liquid film, and
b) it does not take into account the sub-cooling of the liquid film.

These effects can be accounted for by replacing h_{fg} in Eq. 7 by h'_{fg} given by the following Eq.

$$h'_{fg} = h_{fg} + 0.68 \, C_{pL} \, (T_{sat} - T_s)$$

Then, we have, for average heat transfer coefficient for laminar film condensation on a vertical plate:

$$h_{avg} = 0.943 \cdot \left[\frac{\rho_L \, (\rho_L - \rho_V) \cdot k_L^3 \cdot g \cdot h'_{fg}}{\mu_L \cdot L \cdot (T_{sat} - T_s)}\right]^{\frac{1}{4}} \quad \frac{w}{m^2 C} \quad \dots for \;\; 0 < Re_f < 30 \qquad (8)$$

8.2.3 Film Condensation on Inclined Plates, Vertical Tubes, Horizontal Tubes and Horizontal Tube Banks

1- Inclined plates:

Eq. 8 for laminar condensation on vertical plates can also be used for inclined plates. If the plate is inclined at an angle of θ to the vertical, ($\theta \leq 60$ deg.), replacing g by $g.cos(\theta)$ in Eq. 8 gives satisfactory results for laminar condensation on the upper surface of the inclined plate, i.e.

$$h_{\text{inclined}} = 0.943 \cdot \left[\frac{\rho_L \left(\rho_L - \rho_V \right) . k_L^3 . g. cos(\theta) . h'_{fg}}{\mu_L . L . (T_{sat} - T_s)} \right]^{\frac{1}{4}} \qquad 9$$
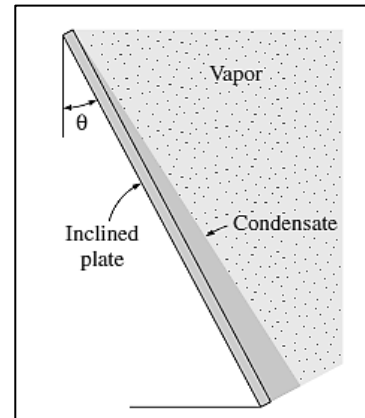
2- Vertical tubes:

Eq. 8 for laminar condensation on vertical plates can also be used to determine heat transfer coefficient for laminar condensation on the outer or inner surface of a vertical tube, if the tube diameter is large compared to the thickness of the liquid film, i.e. if D $\gg \delta$
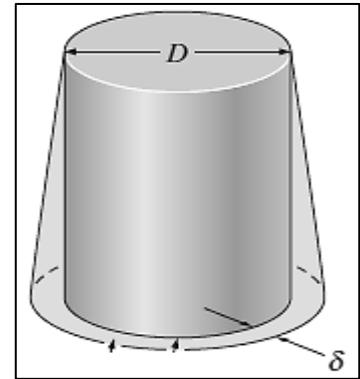
3- Horizontal tube-laminar film condensation:

For laminar film condensation on horizontal tubes and spheres, Nusselt type of analysis gives relations similar to Eq. 8 except that L is replaced by diameter D and the value of the numerical constant is different. We get

$$h_{\text{horiz.}} = 0.729 \cdot \left[\frac{\rho_L \left(\rho_L - \rho_V \right) . k_L^3 . g . h'_{fg}}{\mu_L . D . (T_{sat} - T_s)} \right]^{\frac{1}{4}} \qquad 10$$

It is interesting to compare the laminar condensation on vertical and horizontal tubes. From Eqs. 8 and 10 we can write:

$$\frac{h_{\text{ver.}}}{h_{\text{horiz.}}} = \frac{0.943}{0.729} \cdot \left(\frac{D}{L} \right)^{\frac{1}{4}} = 1.294. \left(\frac{D}{L} \right)^{\frac{1}{4}}$$
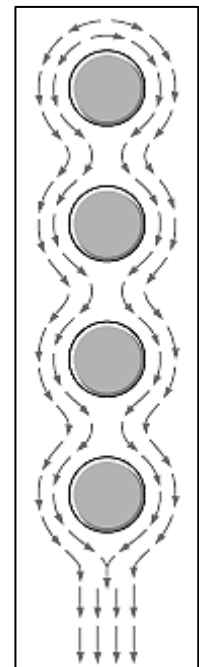
For $h_{\text{ver.}}$ to be equal to $h_{\text{horiz.}}$ we should have:

$$L = (1.294)^4 . D$$
$$L = 2.8 \, D$$

4- Average heat transfer coefficient for film condensation of horizontal tubes on a matrix containing N tubes is obtained by substituting (N.D) in place of D in Eq. 10 for a single horizontal tube, i.e.

$$h_{\text{horiz.}} = 0.729 \cdot \left[\frac{\rho_L \left(\rho_L - \rho_V \right) . k_L^3 . g . h'_{fg}}{\mu_L . N. D . (T_{sat} - T_s)} \right]^{\frac{1}{4}} \qquad 11$$

8.3 BOILING AND EVAPORATION

'Boiling' occurs at the solid-liquid interface when the solid surface is at a temperature T_s, sufficiently above the saturation temperature T_{sat}, of the liquid at that pressure. In contrast, 'evaporation' occurs at the liquid-vapour interface when the vapour pressure above the liquid is less than the saturation pressure of the liquid at the given temperature. Unique feature of the boiling phenomenon is the production of vapour bubbles at the solid-liquid interface causing intense mixing.
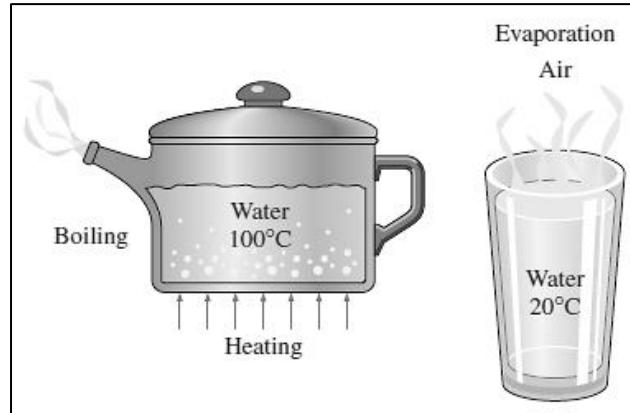
8.3.1 Boiling Modes

Boiling is generally classified as 'pool boiling' and 'flow boiling'.

In pool boiling, there is no bulk fluid flow, and any motion of the fluid is due to natural convection and the movement of bubbles under buoyancy effects. Heating of a liquid by immersing a heating element in it is an example of pool boiling.

When boiling occurs while fluid is in motion under the influence of a pump, it is called flow boiling.

These two modes of boiling are further classified as 'sub-cooled boiling' and 'saturated boiling', in sub-cooled boiling, main body of the liquid is at a temperature below the saturation temperature T_{sat} while in saturated boiling, main body of the liquid is at a temperature equal to T_{sat}. During initial stages of boiling, we have the sub-cooled boiling where bubbles originate at the heating surface, move up due to buoyancy effects, and dissolve in the cooler liquid since the body of the liquid is at a temperature lower than T_{sat}. As the body of the liquid reaches the saturation temperature, bubbles start reaching up to the free surface of the liquid and we say that bulk or saturated boiling is set in motion.

Since boiling is a form of convection heat transfer, boiling heat flux is given by Newton's law of cooling

$$q_{boiling} = h \cdot (T_s - T_{sat}) = h. \Delta T_e \quad W/m^2$$

Where $\Delta T_e = (T_s - T_{sat}) =$ excess temperature.

(a) Subcooled boiling   (b) Saturated boiling
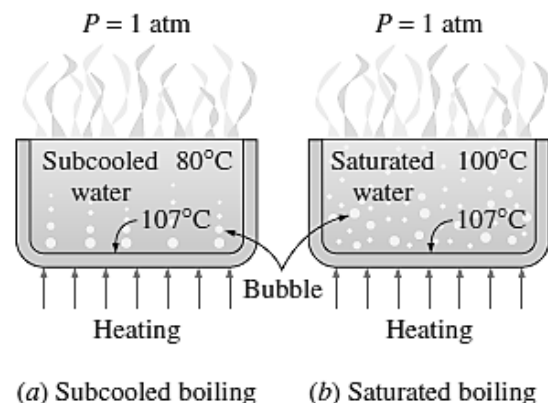
8.3.1 Boiling Regimes and Boiling Curve

Nukiyama performed his experiments on boiling heat transfer in 1934. He used nichrome and platinum wires which were electrically heated while immersed in liquids. In general, four different boiling regimes are observed depending upon the excess temperature (ΔT_e), namely

I.   natural convection boiling (ΔT_e upto about 5 deg.C)
II.   nucleate boiling (ΔT_e from 5 deg to about 30 deg.C)
III. transition boiling (ΔT_e from 30 deg to about 120 deg.C)
IV.   film boiling (ΔT_e beyond 120 deg.C).

Fig. (1) shows a typical boiling curve for water at one atmosphere pressure. General shape of the boiling curve is same for other fluids as well. In Fig. (1), boiling heat flux is plotted against the excess temperature. Also, shape of the boiling curve is independent of the geometry of the heating surface, but depends on the fluid pressure and the specific fluid-heating surface combination.

(I) Natural convection boiling

This range is up to the point A' in Fig. 1. No bubbles are formed up to a small excess temperature of about 5 deg. and the liquid is superheated, rises to the free surface and evaporates from the surface. In this range, the free convection correlations used in the previous chapter can be applied to make heat transfer calculations.
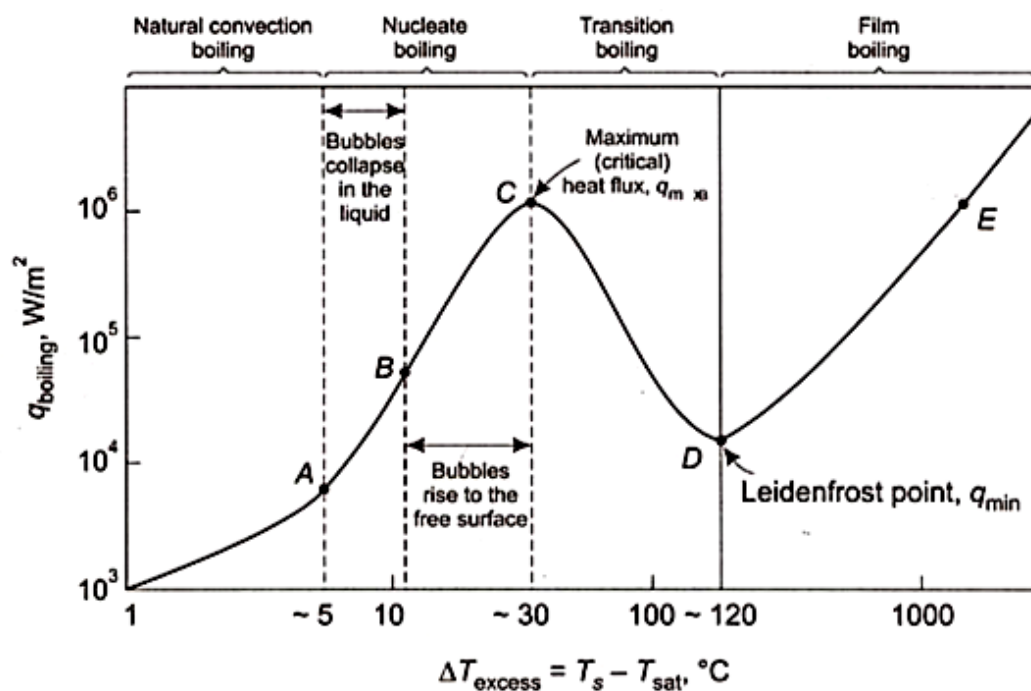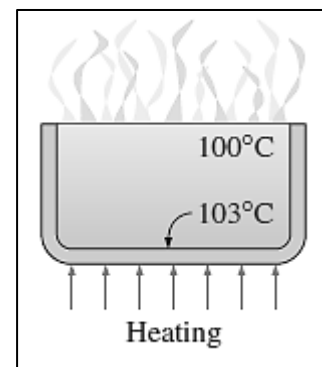

FIGURE 1   Typical boiling curve for water at one atmosphere pressure

(II) Nucleate boiling

Region between 'A' and 'C' is the nucleate boiling region. Starting from point 'A', as ΔT_e increases, bubbles start forming at nucleation sites at an increasing rate. Nucleate boiling region may classified into two sub-regions:

a) region A-B, where the isolated bubbles formed rise up, but do not reach the free surface and collapse in the body of the liquid. Movement of the bubbles through the body of the liquid causes agitation which is responsible for increasing heat transfer in nucleate boiling.

b) region B-C, where the bubbles form at a faster rate at a largely increased number of <u>nucleation sites</u> and rise up in the liquid in almost continuous columns of vapour. These bubbles gush up in the liquid and reach the free surface and then collapse. Heat flux in this region is very large due to this reason. Note at point C there is an inflection in the boiling curve; this is because of the fact that as excess temperature is increased, the heating surface gets almost covered with bubbles and the heat flux increases at a lower rate as ΔT_e, increases, and reaches a maximum at point C. Heat flux at point C is called 'critical' or 'maximum' or 'burnout' heat flux, q_{max}. For water, $q_{max} > 1$ MW/m^2.

It should be clear that from heat transfer point of view, nucleate boiling regime is the most desirable range to operate, since very high heat transfer rates are obtained with relatively small ΔT_e.(under 30°C).
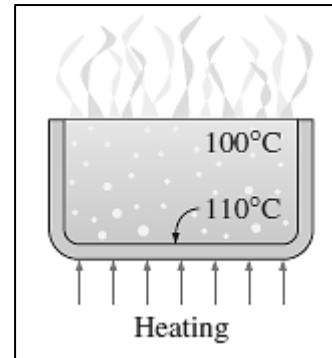
(III) Transition boiling

Region between 'C and 'D' is the transition boiling region. In this range, as the excess temperature increases, the heat flux decreases; this is due to the fact that now a major portion of the heater surface is covered by the vapour film which has a smaller thermal conductivity as compared to that of the liquid, and, therefore, acts as an insulation. At point D, excess temperature is of the order of 120°C.
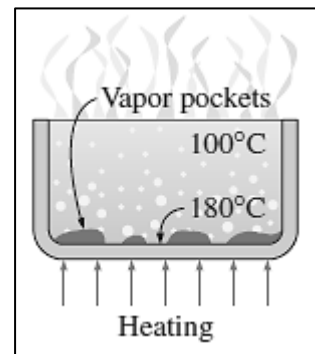
(IV) Film boiling

Region beyond the point D. As excess temperature is further increased, now a stable, vapour completely covers the heater surface. So, at point D, the heat flux reaches a minimum. Now, as the excess temperature is increased further, heat transfer by radiation effect also comes into picture in addition to conduction through the vapour film, and the heat flux increases as shown.
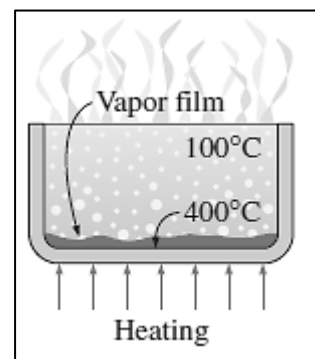
8.3.2 Burnout Phenomenon

In Fig. 1, a continuous boiling curve was shown. However, in practice, when Nukiyama conducted his experiments with an electrically heated nichrome wire immersed in a pool of water, he observed that when a little excess power was supplied to the nichrome wire after reaching point C, wire temperature suddenly increased uncontrollably to the melting point of the wire (i.e.l500 K) and burnout occurred. When the experiment was repeated with platinum

wire (which has a higher melting point of 2045 K), it was possible to maintain heat flux higher than q_{max} without a burnout Now.

Note that the arm C-D of the boiling curve cannot be obtained in the power controlled mode of heating, unless the power applied is reduced suddenly when point C is reached.

As we go on supplying electrical energy to the heater, point C (Fig. 1), i.e. the point of critical or maximum heat flux is reached; now, if we try to go past this point by increasing the heater power, the fluid is not able to accept this increased. and as a result, the heater temperature increases.   that   the   heater   may   melt   or   'burnout'.   Hence,   the   name burnout heat flux' for the heat flux at point C.

Knowledge of 'burnout flux' is very important from practical point of view (in electrically heated surfaces), since any attempt to go past the point C causing a burnout. So, the aim should be to operate at a point as near to the point C as possible.

8.3.3 Simplified relations for boiling heat Transfer with water

Many empirical relations have been developed to estimate the boiling heat-transfer coefficients for water. Some of the simplest relations are those presented by Jakob and Hawkins for water boiling on the outside of submerged surfaces at atmospheric pressure (Table 1).

Surface	$\frac{q}{A}$, kW/m^2	h, W/m$^2 \cdot$°C	Approximate range of ΔT, °C	Approximate range of h, W/m$^2 \cdot$°C
Horizontal	$\frac{q}{A} < 16$	$1042(\Delta T_x)^{1/3}$	0–7.76	0–2060
	$16 < \frac{q}{A} < 240$	$5.56(\Delta T_x)^3$	7.32–14.4	2180–16,600
Vertical	$\frac{q}{A} < 3$	$537(\Delta T_{ax})^{1/7}$	0–4.51	0–670
	$3 < \frac{q}{A} < 63$	$7.96(\Delta T_x)^3$	4.41–9.43	680–6680

Table 1

These heat-transfer coefficients may be modified to take into account the influence of pressure by using the empirical relation

$$h_p = h_1 = \left(\frac{p}{p1}\right)^{0.4}$$

where

hp = heat-transfer coefficient at some pressure p

$h1$ = heat-transfer coefficient at atmospheric pressure as determined from Table 1

p = system pressure

$p1$ = standard atmospheric pressure

For forced-convection boiling inside tubes the following relation is recommended

$$h = 2.54 \cdot (\Delta Te)^3 \cdot e^{p/1.551} \quad W/m^2 C$$

where ΔTe is the temperature difference between the surface and saturated liquid in degrees Celsius and p is the pressure in MPa. The heat-transfer coefficient has the units of watts per square meter per degree Celsius. The equation is valid over a pressure range of 5 to 170 atm.

- CHAPTER NINE-

HEAT EXCHANGER

9.1 Introduction

9.1.2 Heat Exchanger Applications

9.1.3 Types of Heat Exchangers

9.2 The Overall Heat Transfer Coefficient

9.3 Fouling Factors

9.4 The LMTD Method for Heat Exchanger Analysis

9.4.2 Correction Factors for Multi-Pass and Cross-Flow HX

9.5 The Effectiveness NTU Method for Heat Exchanger Analysis

9.5.2 Effectiveness-NTU Relation for a Parallel-Flow Heat Exchanger

9.5.2 Effectiveness-NTU Relation for a Counter-Flow Heat Exchanger

9.6 Selection of Heat Exchangers

HEAT EXCHANGER

9.1 Introduction

'Heat exchanger' is one of the most commonly used process equipment in industry and research. Function of a heat exchanger is to transfer energy; this transfer of energy may occur to a single fluid (as the case of a boilers where heat is transferred to water) or between two fluids that are at different temperatures (as in the case of an automobile radiator where heat is transferred from hot water to air). In some cases, there are more than two streams of fluid exchanging heat in a heat exchanger. Heat exchangers of several designs in a variety of size varying from 'miniature' to 'huge'

9.1.2 Some typical examples of heat exchanger applications are:

(i) Thermal power plants (boilers, superheaters, steam condensers, etc.)
(ii) Chemical process industry (variety of heat exchangers between different types of fluids, in reactors)
(iii) Refrigeration and air-conditioning (evaporators, condensers, coolers)
(iv) Automobile industry (radiators, all engine cooling and fuel cooling arrangements)
(v) Cryogenic industry
(vi) Research

9.1.3 Types of Heat Exchangers

Heat exchangers may be classified in several ways:
(i) According to heat exchange process
(ii) According to relative direction of flow of hot and cold fluids
(iii) According to constructional features, compactness, etc.
(iv) According to the state of the fluid in the heat exchanger.

(i) Classification according to heat exchange process:
Heat exchangers may be of **'direct contact type'** or **'indirect contact type'**. In direct contact type, the two fluids come in direct contact with each other and exchange heat, e.g. air and water exchanging heat in a cooling tower.

Indirect contact type can be further classified as **'recuperators'** and **'regenerators'**. Recuperators are most commonly used; here, the hot and cold fluids separated from each other by a solid wall and heat is transferred from one fluid to the other across this wall. Regenerators, also called periodic flow heat exchangers', hot and cold streams alternately flow through a solid matrix during the 'hot blow', the matrix stores the heat given up by the hot stream and during 'cold blow', the stored heat' is given up by the solid matrix to the cold stream.

(ii) Classification according to relative direction of hot and cold fluids:
If the hot and cold fluids flow parallel to each other, it is known as **'parallel flow'** heat exchanger; if the two fluids flow opposite to each other, it is of **counter-flow**' type. If the fluids flow perpendicular to each other, then, we have **'cross flow'** type of heat exchanger. These three types of heat exchangers are shown schematically in Fig. 1.

(iii) Classification according to constructional features:
Basically, there are three types: **(a) concentric tubes type (b) shell and tube type, and (c) compact heat exchangers.**

In concentric tubes type of heat exchanger, one tube is located inside another; one fluid flows through the inside tube and the other fluid flows in the annular space between the tubes.

Shell and tube type of heat exchanger (see fig 2) is very popular in industry because of its reliability and high heat transfer effectiveness. Here, one of the fluids flows within a bundle of tubes placed within a shell. And, the other fluid flows through the shell over the surfaces of the tubes.

Compact heat exchangers are special purpose heat exchangers which provide very high surface area per cubic metre of volume, known as 'area density'. These are generally used for gases, since usually gas side heat transfer coefficient is small and therefore, it is needed to provide larger areas.

(iv) Classification according to state of the fluid

In all the types of heat exchangers discussed above, both the fluids changed their temperature along the length of heat exchanger. But, this need not be the case always. A heat exchanger may be used to condense a fluid in which case the condensing fluid will be at a constant temperature throughout the length of the heat exchanger, while the other (cold) fluid will increase in temperature as it passes through the heat exchanger, absorbing the latent heat of condensation released by the condensing fluid. Such a heat exchanger is called a **'Condenser'**. If, on the other hand, one of the fluids evaporates in a heat exchanger, such a heat exchanger is called an **Evaporator**.

Figure 1

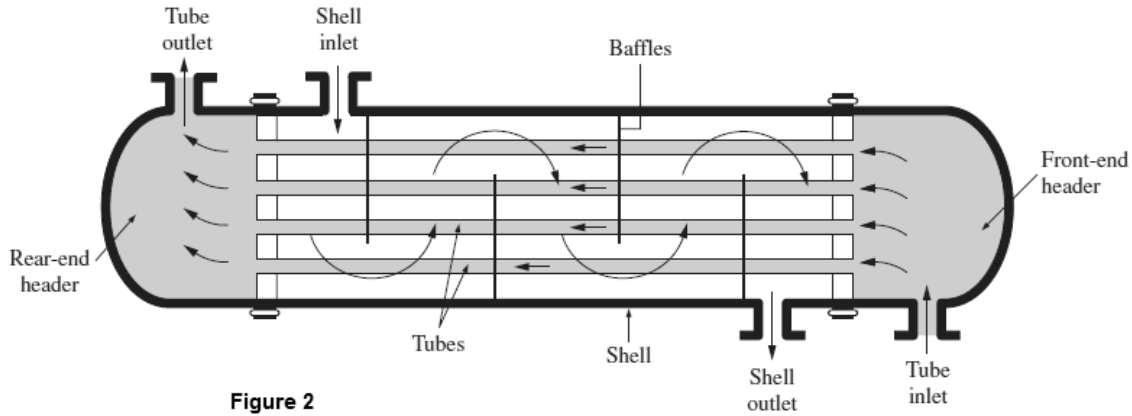(a) Parallel flow   (b) Counter flow   (C) cross flow
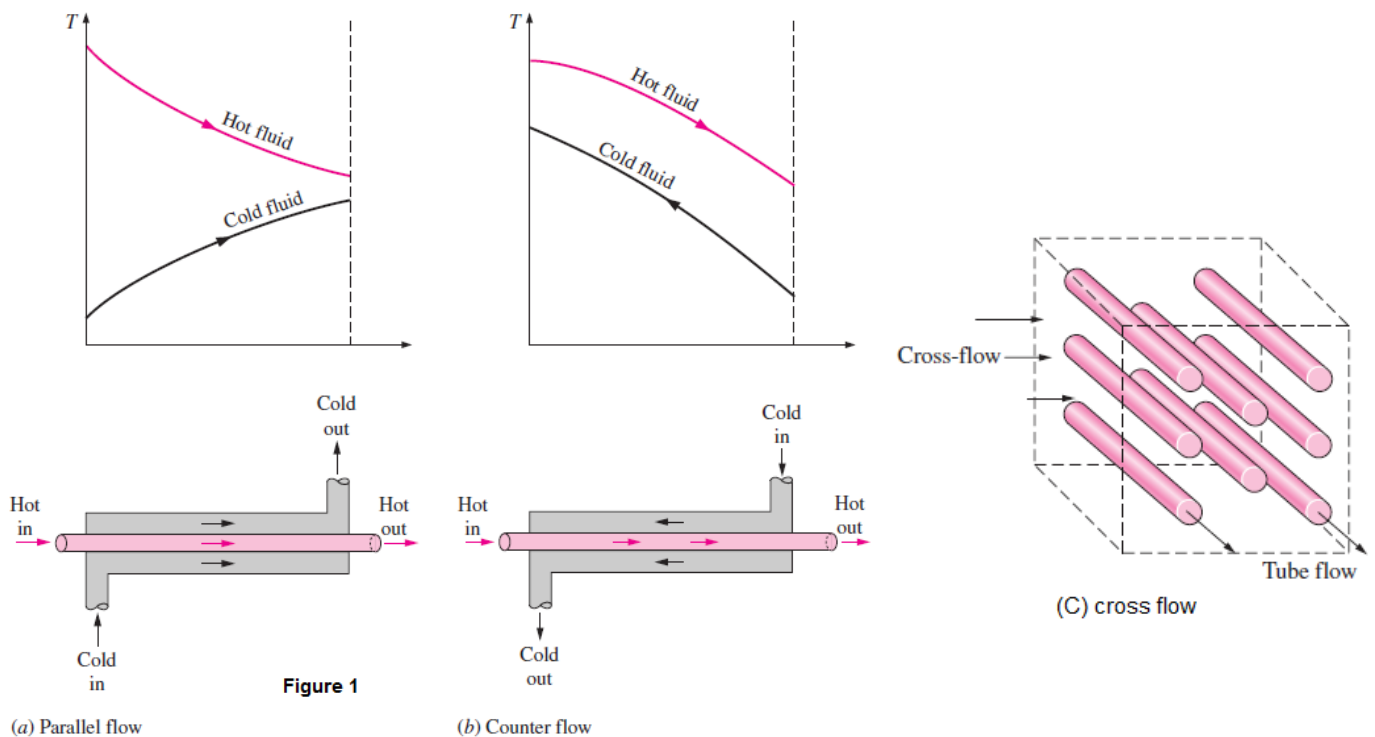
Figure 2

9.2 THE OVERALL HEAT TRANSFER COEFFICIENT

A heat exchanger typically involves two flowing fluids separated by a solid wall. Heat is first transferred from the hot fluid to the wall by *convection,* through the wall by *conduction,* and from the wall to the cold fluid again by *convection.* The thermal resistance network associated with this heat transfer process involves two convection and one conduction resistances, as shown in Figure 3.

$$R = R_{tot} = R_i + R_{wall} + R_o = \frac{1}{h_i A_i} + \frac{\ln(D_o/D_i)}{2\pi k L} + \frac{1}{h_o A_o}$$

$$Q = \frac{\Delta T}{R} = U A \Delta T$$

Where U is the **overall heat transfer coefficient,** whose unit is W/m²°C. Which is identical to the unit of the convection coefficient h. canceling ΔT the Eq. reduces to

$$R = \frac{1}{U_o A_o} = \frac{1}{U_i A_i} = \frac{1}{U A_s} = \frac{1}{h_i A_i} + \frac{\ln(D_o/D_i)}{2\pi k L} + \frac{1}{h_o A_o}$$

When the wall thickness of the tube is small and the thermal conductivity of the tube material is high, as is usually the case, the thermal resistance of the tube is negligible (R_{wall} =0) and the inner and outer surfaces of the tube are almost identical ($Ai = Ao = As$). Then the Eq. of the overall heat transfer coefficient simplifies to

$$\frac{1}{U} = \frac{1}{h_i} + \frac{1}{h_o}$$

Figure 3

The overall heat transfer coefficient U in this Eq. is dominated by the *smaller* convection coefficient, since the inverse of a large number is small. When one of the convection coefficients is *much smaller* than the other (say, $hi \ll ho$), we have $1/hi \gg 1/ho$, and thus $U \cong hi$. Therefore, the smaller heat transfer coefficient creates a *bottleneck* on the path of heat flow. This situation arises frequently when one of the fluids is a gas and the other is a liquid. In such cases, fins are commonly used on the gas side to enhance the product UA_s and thus the heat transfer on that side.

When the tube is *finned* on one side to enhance heat transfer, the total heat transfer surface area on the finned side becomes

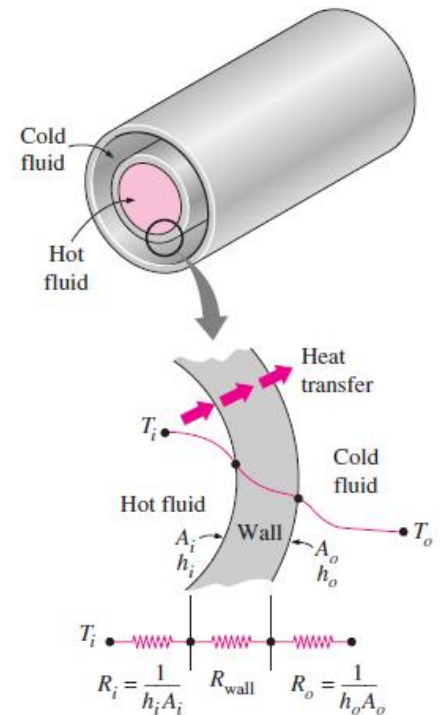$$A_s = A_{unfinned} + \eta_{fin} A_{fin}$$

Where η_{fin} is the fin efficiency. This way, the temperature drop along the fins is accounted.

$$R = \frac{1}{U_o A_o} = \frac{1}{U_i A_i} = \frac{1}{U A_s} = \frac{1}{h_i A_i} + \frac{1}{h_o (A_{unfinned} + \eta_{fin} A_{fin})_o}$$

This Eq. is an example for outer finned tube, neglected thermal resistance of the tube.

TABLE 1

Representative values of the overall heat transfer coefficients in heat exchangers

Type of heat exchanger	U, W/m² · °C*
Water-to-water	850–1700
Water-to-oil	100–350
Water-to-gasoline or kerosene	300–1000
Feedwater heaters	1000–8500
Steam-to-light fuel oil	200–400
Steam-to-heavy fuel oil	50–200
Steam condenser	1000–6000
Freon condenser (water cooled)	300–1000
Ammonia condenser (water cooled)	800–1400
Alcohol condensers (water cooled)	250–700
Gas-to-gas	10–40
Water-to-air in finned tubes (water in tubes)	30–60†
	400–850†
Steam-to-air in finned tubes (steam in tubes)	30–300†
	400–4000‡

9.3 Fouling factors

Note that: above analysis was for clean heat transfer surfaces. However, with passage of time, the surfaces become 'dirty' because of scaling, deposits, corrosion, etc. This results in a reduction in heat transfer coefficient since the scale offers a thermal resistance to heat transfer. Fouling may be categorized as follows:

(i) Due to scaling or precipitation.
(ii) Due to deposits of divided particulates.
(iii) Due to chemical reaction.
(iv) Due to corrosion.
(v) Due to attachments of algae or other biological materials.
(vi) Due to crystallization on the surface by subcooling.

Effect of fouling is accounted for by a term called, 'Fouling factor',

$$R_f = \frac{1}{U_{dirty}} - \frac{1}{U_{clean}}$$

R_f is zero for a new heat exchanger.

While taking into account the effect of fouling, the 'fouling resistance' ($=R_f$/area) should be added to the other thermal resistances. For example, for a tube, we can write:

$$R = \frac{1}{U_o A_o} = \frac{1}{U_i A_i} = \frac{1}{U A_s} = \frac{1}{h_i A_i} + \frac{R_{fi}}{A_i} + \frac{\ln(D_o/D_i)}{2\pi k L} + \frac{R_{fo}}{A_o} + \frac{1}{h_o A_o}$$

where, R_{fi} and R_{fo} are the fouling factors for the inside and outside surfaces, respectively,

Note: the effect of fouling has to be included in the design stage of heat exchanger.

Fouling factor depends on flow velocity and operating temperature; fouling increases with decreasing velocity and increasing temperature. Based on experience Tubular Exchanger Manufacturers' Association (TEMA) has given suggested values of fouling factors. Some of these values are given in Table 2

Table 2

Fluid	R_f (m²C/W)
LIQUIDS:	
Fuel oil	0.00088
Quench oil	0.0007
Transformer oil	0.00018
Hydraulic fluid	0.000238
Molten salts	0.000119
Industrial organic heat transfer media	0.000119
Refrigerant liquids	0.00018
Caustic solutions	0.000476
Vegetable oils	0.000715
Gasoline, naptha, light distillates, kerosene	0.000238
Light gas oil	0.000476
Heavy gas oil	0.000715
GASES & VAPOURS:	
Solvent vapours	0.000238
Acid gases	0.000238
Natural gas	0.000238
Air	0.000119 – 0.000238
Flue gases	0.000238 – 0.000715
Steam (sat., oil free)	0.000119 – 0.000357
WATER:	
River water, sea water, distilled water, boiler feed water:	
Below 50 deg.C	0.0001
Above 50 deg.C	0.0002

9.4 The LMTD Method for Heat Exchanger Analysis

1- Parallel Flow Heat Exchanger

Consider a double pipe, parallel flow heat exchanger, in which a hot fluid and a cold fluid flow parallel to each other, separated by a solid wall. Hot fluid enters at a temperature of T_{h1} and leaves the heat exchanger at a temperature of T_{h2}; cold fluid enters the heat exchanger at a temperature of T_{c1}, and leaves at a temperature of T_{c2}. This situation is shown in Fig. 4 .
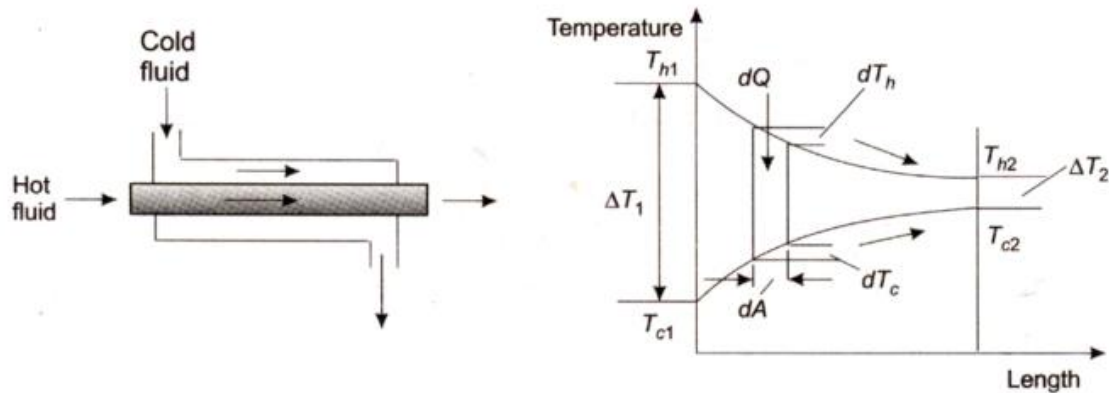
FIGURE 4 Parallel flow heat exchanger

We desire to get an expression for the rate of heat transfer in this heat exchanger in the following form:

$$Q = U.A.\Delta T_m$$

Where,

U = overall heat transfer coefficient

A = area for heat transfer (should be the same area on which U is based)

ΔT_m = a mean temperature difference between the fluids.

Now, we make the following assumptions:

(i) U is considered as a constant throughout the length (or area) of the heat exchanger

(ii) Properties of fluids (such as specific heat) are also considered to be constant with temperature

(iii) Heat exchange takes place only between the two fluids and there is no loss of heat to the surroundings, i.e. perfect insulation of heat exchanger is assumed

(iv) Changes in potential and kinetic energy are negligible

Area 'A' is constant for a given heat exchanger. However, we see from Fig. 4 that the temperatures of the two fluids vary along the length (or area) of the heat exchanger, i.e. the temperature difference between the hot and cold fluids is not a constant along the length of the heat exchanger, but varies along the length.

Consider an elemental area dA of the heat exchanger. Then, by applying the First law, we can write:

Heat given up by the hot fluid = heat received by the cold fluid.

i.e.

$$dQ = -m_h.C_{ph}.dT_h = m_c.C_{pc}.dT_c \qquad\qquad 1$$

Here, the temperature of hot fluid decreases as the length increases. So, a negative sign is put in front of $m_h.C_{ph}.dT_h$

Now, dQ for the elemental area dA, can also be expressed as:

$$dQ = U.(T_h - T_c).dA \qquad 2$$

Now, from Eq. 1, we have:

$$dT_h = \frac{-dQ}{m_h.C_{ph}}$$

$$dT_c = \frac{dQ}{m_c.C_{pc}}$$

So

$$dT_h - dT_c = d(T_h - T_c) = -dQ\left[\frac{1}{m_h.C_{ph}} + \frac{1}{m_c.C_{pc}}\right] \qquad 3$$

Substituting for dQ from Eq. 2, we get:

$$d(T_h - T_c) = -U.(T_h - T_c).dA\left[\frac{1}{m_h.C_{ph}} + \frac{1}{m_c.C_{pc}}\right]$$

$$\frac{d(T_h - T_c)}{(T_h - T_c)} = -U\left[\frac{1}{m_h.C_{ph}} + \frac{1}{m_c.C_{pc}}\right].dA \qquad 4$$

Integrating Eq. 4 between the inlet and exit of the heat exchanger (i.e. between conditions 1 and 2):

$$ln\left[\frac{T_{h2} - T_{c2}}{T_{h1} - T_{c1}}\right] = -U.A\left[\frac{1}{m_h.C_{ph}} + \frac{1}{m_c.C_{pc}}\right] \qquad 5$$

Now, considering the total heat transfer rate for the entire heat exchanger, we have:

$$m_h.C_{ph} = \frac{Q}{T_{h1} - T_{h2}}$$

$$m_c.C_{pc} = \frac{Q}{T_{c2} - T_{c1}}$$

Sub in Eq. 5

$$ln\left[\frac{T_{h2} - T_{c2}}{T_{h1} - T_{c1}}\right] = \frac{-U.A}{Q}(T_{h1} - T_{h2} + T_{c2} - T_{c1})$$

$$Q = U.A.\frac{(T_{h2} - T_{c2}) - (T_{h1} - T_{c1})}{ln\left[\frac{T_{h2} - T_{c2}}{T_{h1} - T_{c1}}\right]} \qquad 6$$

Now consider this Eq. $Q = U.A.\Delta T_m$ we observe that:

$$\Delta T_m = \frac{(T_{h2} - T_{c2}) - (T_{h1} - T_{c1})}{ln\left[\frac{T_{h2} - T_{c2}}{T_{h1} - T_{c1}}\right]}$$

Since this mean temperature difference varies in a logarithmic manner, it is called 'Logarithmic Mean Temperature Difference' or, simply LMTD.
So, we write:

$$LMTD = \frac{(T_{h2} - T_{c2}) - (T_{h1} - T_{c1})}{ln\left[\frac{T_{h2} - T_{c2}}{T_{h1} - T_{c1}}\right]} \qquad 7$$

Now, note that $(T_{h2} - T_{c2})$ is the temperature difference at the exit and $(T_{h1} - T_{c1})$ is the temperature difference at the inlet of the heat exchanger see fig. (4). If we denote the temperature differences at the inlet and exit of the heat exchanger by ΔT_1 and ΔT_2, respectively, we can write:

$$LMTD = \frac{(\Delta T_2 - \Delta T_1)}{ln\left[\frac{\Delta T_2}{\Delta T_1}\right]} = \frac{(\Delta T_1 - \Delta T_2)}{ln\left[\frac{\Delta T_1}{\Delta T_2}\right]} \qquad 8$$

2- Counter flow heat exchanger

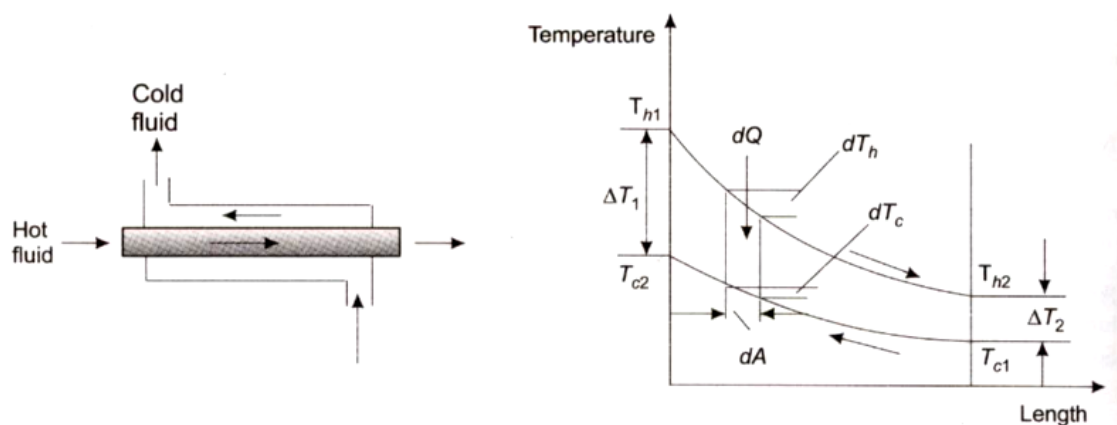For Counter flow the same of Eq. 8 will be used

But here:

$$\Delta T_1 = T_{h2} - T_{c1}$$

$$\Delta T_2 = T_{h1} - T_{c2}$$

See fig 5.

So

$$LMTD = \frac{(T_{h2} - T_{c1}) - (T_{h1} - T_{c2})}{ln\left[\frac{T_{h2} - T_{c1}}{T_{h1} - T_{c2}}\right]}$$

FIGURE 5   Counter-flow heat exchanger

Comments:

1- LMTD for a counter flow heat exchanger is always greater than that for a parallel flow heat exchanger. This means that to transfer the same amount of heat, counter flow unit will require a smaller heat transfer surface as compared to a parallel flow unit. This is the reason why a counter-flow heat exchanger is usually preferred.

2- LMTD can easily be calculated when all the end temperatures of the fluids are known. Then, immediately, the heat transfer rate is determined from the Eq. $Q = U.A.(LMTD)$.

3- The term $m.C_p$ is generally replaced by C in heat exchanger analysis, i.e. $C = m.C_p$ Here, C is known as 'heat capacity rate' or, simply 'capacity rate' of that particular fluid. Thus, the capacity rates for hot and cold fluids are:

$$C_h = m_h.C_{ph} \quad W/C$$
$$C_c = m_c.C_{pc} \quad W/C$$

Then, the heat transfer rate is given by:

$$Q = C_h.(T_{h1} - T_{h2}) \quad W$$
$$Q = C_c.(T_{c2} - T_{c1}) \quad W$$

i.e. to transfer a given amount of heat, higher the heat capacity rate of a fluid, lower will be the temperature rise (or fall) of that particular fluid. If the heat capacity rates of both the hot and cold fluids are equal, then, the total temperature drop of the hot fluid will be equal to the total temperature rise of the cold fluid. See Fig. 6
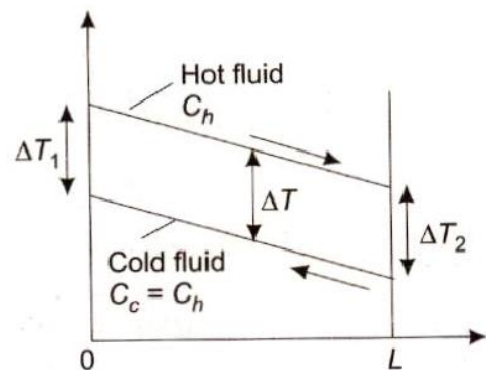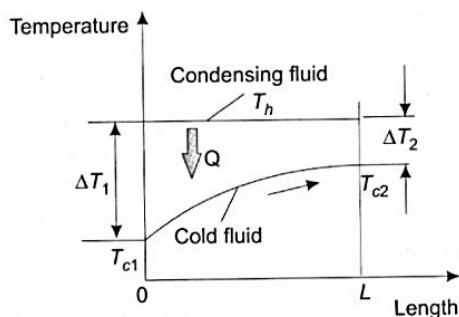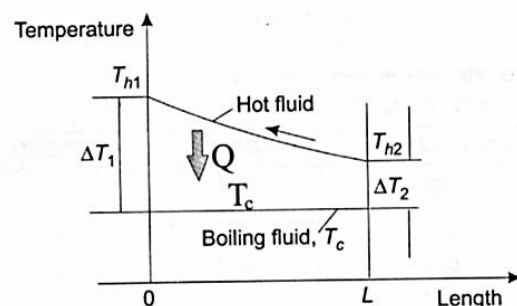

FIGURE 6

4- When a fluid is condensing or boiling, its temperature is essentially constant, i.e. $T_{h1} = T_{h2}$ for a condensing fluid and $T_{c2} = T_{c1}$ for a boiling liquid. In other words, ΔT for the condensing or boiling fluid is zero. But since a finite amount of heat is transferred, $(= m.h_{fg})$ we say that capacity rate of a condensing or boiling fluid tends to infinity. Temperature profiles for fluids in a heat exchanger when one of the fluids is condensing or boiling are shown in Fig. 7 (b) and (c), respectively. LMTD for both these cases is determined by the same procedure as for the parallel or counter-flow heat exchangers

(b) One of the fluids condensing

(c) One of the fluids boiling

Figure 7

9.4.2 Correction Factors for Multi-pass and Cross-flow Heat Exchangers

LMTD relations derived above are applicable to parallel flow and counter flow heat exchangers only. But, in practice, cross-flow heat exchangers (e.g. automobile radiators) and shell-and-tube heat exchangers, with more than one pass in tube side, are also used. In such cases, first, LMTD is calculated as if for a counter-flow heat exchanger with the inlet and exit temperatures for the two fluids as per the actual data, and next, a 'correction factor (F)' is applied to the calculated LMTD to get the mean temperature difference between the fluids. Now, heat transfer rate is calculated as:

$$Q = U.A.(F.LMTD) \, W$$

A value of correction factor (F) for a selected heat exchanger (as an example) is given in graphical representation in Fig.8. Correction factor F is plotted as function of two parameters, i.e. P and R, defined as:
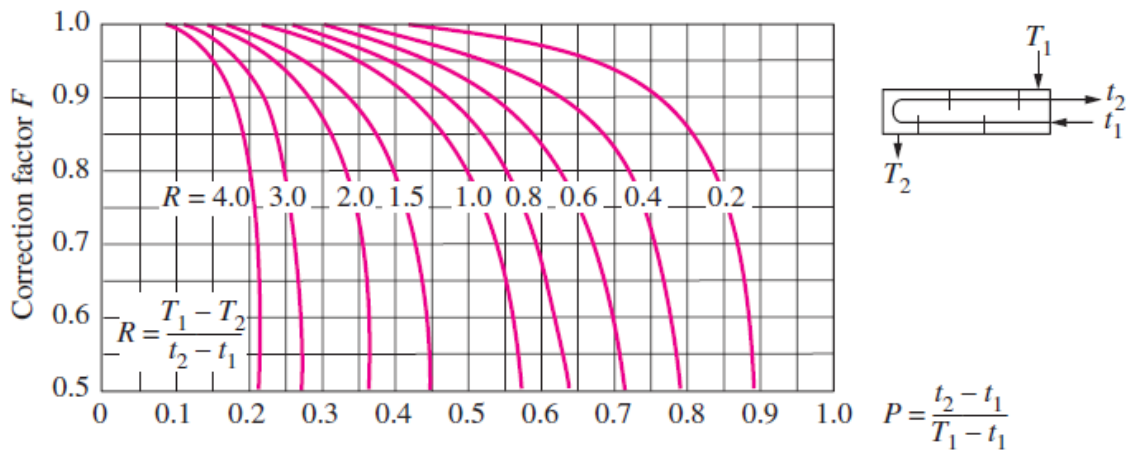

Figure 8 One-shell pass and 2, 4, 6, etc. (any multiple of 2), tube passes

Note: for a condenser or boiler, F = 1, irrespective of the configuration of the heat exchanger.

9.5 The Effectiveness NTU Method for Heat Exchanger Analysis

LMTD can readily be determined when all the four end temperatures are either given, or can easily be calculated. Then, the area required, A (i.e. the size of the HX) is easily found out by applying the equation: $Q = U.A.(LMTD)$. In other words, LMTD method is very convenient to use for sizing problems. However, there are certain problems where only the inlet temperatures of both the fluids are specified, along with the flow rates and the overall heat transfer coefficients, and the heat transfer rate and the exit temperatures of the fluids are to be calculated. The Effectiveness-NTU method, developed by Kays and London in 1955, overcomes this problem and makes the solution straight forward. Effectiveness-NTU method is also useful in solving heat exchanger problems, where off-design conditions exist; i.e. for example, the heat exchanger might have been designed for some particular flow rates of fluids; now, to find out what happens to the performance if flow rate of one of the fluids is reduced to, say, 75 % of the design flow rate, and so on.

Before we develop the Effectiveness-NTU relations for different types of heat exchangers, let us define a few quantities:

- Effectiveness of a heat exchanger (ε):

$$\varepsilon = \frac{Q}{Q_{max}}$$

Where,

Q = actual heat transferred in the heat exchanger.

Q_{max} = maximum possible heat transfer in the heat exchanger.

Now, actual heat transfer rate in a heat exchanger is given by:

$$Q = m_h.C_{ph}.(T_{h1} - T_{h2}) = C_h.(T_{h1} - T_{h2})$$
$$Q = m_c.C_{pc}.(T_{c2} - T_{c1}) = C_c.(T_{c2} - T_{c1})$$

Now, C_h may be equal to C_c or less than C_c or greater than C_c

If $C_h < C_c$ we designate C_h as C_{min};

Instead, if $C_h > C_c$ we designate C_c as C_{min}.

And in each case, capacity rate of the other fluid is designated as C_{max}

Capacity ratio is defined as:

$$C = \frac{C_{min}}{C_{max}}$$

- Maximum possible heat transfer in a heat exchanger (Q_{max}):

Now, consider a heat exchanger where the hot fluid is cooled from a temperature of T_{h1} to T_{h2} and the cold fluid heated from T_{c1} to T_{c2}. So, the maximum temperature differential in the heat exchanger is (T_{h1} -T_{c1}). Now, if the heat exchanger had an infinite area, the hot fluid will be cooled from T_{h1} to T_{c1} or the cold fluid may be heated from T_{c1} to T_{h1} However, which fluid will experience the maximum temperature differential (T_{h1} -T_{c1}) will depend upon which fluid has the minimum capacity rate.

If hot fluid has the minimum capacity rate, we can write:

$$Q_{max} = C_h.(T_{h1} - T_{c1})$$

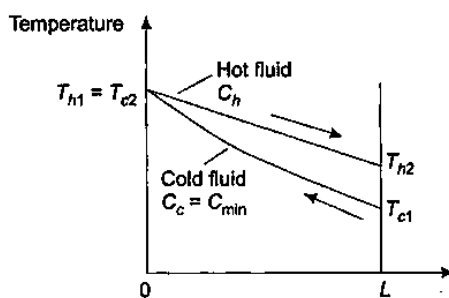Instead, if cold fluid has the minimum capacity rate, we write:
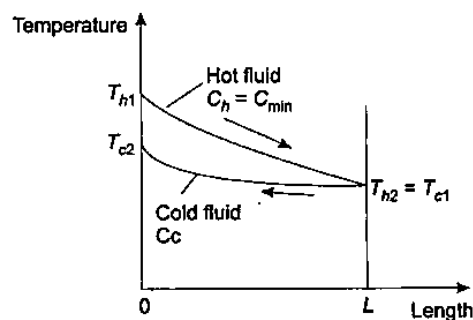
$$Q_{max} = C_c.(T_{h1} - T_{c1})$$

Or, more generally, we write:

$$Q_{max} = C_{min}.(T_{h1} - T_{c1})$$

These situations are represented graphically in Fig. 9:

FIGURE 9(a) Cold fluid has minimum capacity rate

FIGURE 9(b) Hot fluid has minimum capacity rate

Therefore, we can write for effectiveness:

$$\varepsilon = \frac{Q}{Q_{max}} = \frac{C_h.(T_{h1} - T_{h2})}{C_{min}.(T_{h1} - T_{c1})} = \frac{C_c.(T_{c2} - T_{c1})}{C_{min}.(T_{h1} - T_{c1})} \qquad 9$$

Now, if hot fluid is the 'minimum fluid' (i.e. $C_h < C_c$), the effectiveness:

$$\varepsilon = \frac{(T_{h1} - T_{h2})}{(T_{h1} - T_{c1})}$$

And, if cold fluid is the 'minimum fluid' (i.e. $C_c < C_h$), the effectiveness:

$$\varepsilon = \frac{(T_{h1} - T_{h2})}{(T_{h1} - T_{c1})}$$

- **Number of Transfer Units (NTU)** (which is a dimensionless number), is defined as:

$$NTU = \frac{U.A}{C_{min}}$$

Now, for any heat exchanger, effectiveness can be expressed as a function of the NTU and capacity ratio

$$\varepsilon = f\left(NTU, \frac{C_{min}}{C_{max}}\right)$$

9.5.2 Effectiveness-NTU Relation for a Parallel-flow Heat Exchanger
Consider the parallel-flow heat exchanger shown in Fig. 4. Assumptions for this derivation remain the same as for the LMTD method.
Continuing from Eq. 5:

$$ln\left[\frac{T_{h2} - T_{c2}}{T_{h1} - T_{c1}}\right] = -U.A\left[\frac{1}{m_h.C_{ph}} + \frac{1}{m_c.C_{pc}}\right] \qquad 5$$

Now, one of the two fluids is the 'minimum' fluid and the other is the 'maximum' fluid. Whichever may be the minimum fluid, we can write Eq. 5 as:

$$ln\left[\frac{T_{h2} - T_{c2}}{T_{h1} - T_{c1}}\right] = \frac{-U.A}{C_{min}}.\left[1 + \frac{C_{min}}{C_{max}}\right]$$

$$\frac{T_{h2} - T_{c2}}{T_{h1} - T_{c1}} = exp\left[-NTU.\left[1 + \frac{C_{min}}{C_{max}}\right]\right]$$

Now, substituting for T_{h2} and T_{c2} from Eq. 9, we get:

$$\frac{\left[T_{h1} - \varepsilon.\frac{C_{min}}{C_h}.(T_{h1} - T_{c1})\right] - \left[T_{c1} + \varepsilon.\frac{C_{min}}{C_c}.(T_{h1} - T_{c1})\right]}{T_{h1} - T_{c1}} = exp\left[-NTU.\left[1 + \frac{C_{min}}{C_{max}}\right]\right]$$

$$\frac{(T_{h1} - T_{c1}) - \varepsilon.C_{min}.(T_{h1} - T_{c1}).\left[\frac{1}{C_h} + \frac{1}{C_c}\right]}{T_{h1} - T_{c1}} = exp\left[-NTU.\left[1 + \frac{C_{min}}{C_{max}}\right]\right]$$

$$1 - \varepsilon.C_{min}.\left[\frac{1}{C_h} + \frac{1}{C_c}\right] = exp\left[-NTU.\left[1 + \frac{C_{min}}{C_{max}}\right]\right]$$

Now, assuming $C_h > C_c$ i.e. cold fluid as the 'minimum fluid', we have: $C_{min} = C_c$ and $C_{max} = C_h$

Therefore,

$$1 - \varepsilon.\left[1 + \frac{C_{min}}{C_{max}}\right] = exp\left[-NTU.\left[1 + \frac{C_{min}}{C_{max}}\right]\right]$$

$$\varepsilon = \frac{1 - exp\left[-NTU.\left[1 + \frac{C_{min}}{C_{max}}\right]\right]}{1 + \frac{C_{min}}{C_{max}}} \qquad 10$$

$$\varepsilon = \frac{1 - exp[-NTU.[1 + C]]}{1 + C} \qquad 11$$

Where $C = \frac{C_{min}}{C_{max}}$

Special cases:
(i) For a condenser or boiler i.e. one of the fluids undergoes a phase change. Therefore, $C_{max} \rightarrow \infty$ i.e. Capacity ratio, C = 0. Then effectiveness relation (for all heat exchangers) reduces to:

$$\varepsilon = 1 - exp[-NTU] \qquad 12$$

(ii) When C = 1, i.e. $C_{min} = C_{max}$. This is the case of a typical, gas turbine regenerator. In this case,

$$\varepsilon = \frac{1 - exp[-2.NTU]}{2} \qquad 13$$

9.5.2 Effectiveness-NTU Relation for a Counter-flow Heat Exchanger
Again, consider the counter-flow heat exchanger shown in Fig. 5. The Effectiveness-NTU Relation for this HX will be

$$\varepsilon = \frac{1 - exp[-NTU.[1 - C]]}{1 - C.exp[-NTU.[1 - C]]}$$

TABLE 3

Effectiveness relations for heat exchangers: NTU = UA_s/C_{min} and $c = C_{min}/C_{max} = (\dot{m}C_p)_{min}/(\dot{m}C_p)_{max}$ (Kays and London, Ref. 5.)

Heat exchanger type	Effectiveness relation
1 Double pipe:	
Parallel-flow	$\varepsilon = \dfrac{1 - \exp\,[-NTU(1 + c)]}{1 + c}$
Counter-flow	$\varepsilon = \dfrac{1 - \exp\,[-NTU(1 - c)]}{1 - c\exp\,[-NTU(1 - c)]}$
2 Shell and tube:	
One-shell pass 2, 4, . . . tube passes	$\varepsilon = 2\left\{1 + c + \sqrt{1 + c^2}\,\dfrac{1 + \exp\,[-NTU\sqrt{1 + c^2}]}{1 - \exp\,[-NTU\sqrt{1 + c^2}]}\right\}^{-1}$
3 Cross-flow (single-pass)	
Both fluids unmixed	$\varepsilon = 1 - \exp\left\{\dfrac{NTU^{0.22}}{c}\,[\exp\,(-c\,NTU^{0.78}) - 1]\right\}$
C_{max} mixed, C_{min} unmixed	$\varepsilon = \dfrac{1}{c}(1 - \exp\,\{1 - c[1 - \exp\,(-NTU)]\})$
C_{min} mixed, C_{max} unmixed	$\varepsilon = 1 - \exp\left\{-\dfrac{1}{c}[1 - \exp\,(-c\,NTU)]\right\}$
4 All heat exchangers with $c = 0$	$\varepsilon = 1 - \exp(-NTU)$

9.6 SELECTION OF HEAT EXCHANGERS

Heat exchangers are complicated devices, and the results obtained with the simplified approaches presented above should be used with care. For example, we assumed that the overall heat transfer coefficient U is constant throughout the heat exchanger and that the convection heat transfer coefficients can be predicted using the convection correlations. However, it should be kept in mind that the uncertainty in the predicted value of U can even exceed 30 percent.

Thus, it is natural to tend to overdesign the heat exchangers in order to avoid unpleasant surprises.

Engineers in industry often find themselves in a position to select heat exchangers to accomplish certain heat transfer tasks. Usually, the goal is to heat or cool a certain fluid at a known mass flow rate and temperature to a desired temperature. Thus, the rate of heat transfer in the heat exchanger is

$$Q = m^{.}Cp(T_{in} - T_{out})$$

This gives the heat transfer requirement of the heat exchanger before having any idea about the heat exchanger itself.

An engineer going through catalogs of heat exchanger manufacturers will be surprised by the type and number of readily available heat exchangers. The proper selection depends on several factors.

1. **Heat Transfer Rate**
   This is the most important quantity in the selection of a heat exchanger. A heat exchanger should be capable of transferring heat at the specified rate in order to achieve the desired temperature change of the fluid at the specified mass flow rate.

2. **Cost**
   Budgetary limitations usually play an important role in the selection of heat exchangers; an off-the-shelf heat exchanger has a cost advantage over those made to order. The operation and maintenance costs of the heat exchanger are also important considerations in assessing the overall cost.

3. **Pumping Power**
   In a heat exchanger, both fluids are usually forced to flow by pumps or fans that consume electrical power. The annual cost of electricity associated with the operation of the pumps and fans can be determined from

   $$Operating\ cost = (Pumping\ power, kW) * (Hours\ of\ operation, h)$$
   $$* (Price\ of\ electricity, \$/kWh)$$

   Where the pumping power is the total electrical power consumed by the motors of the pumps and fans. For example, a heat exchanger that involves a 1-hp pump and a $\frac{1}{3}$ -hp fan (1 hp = 0.746 kW) operating 8 h a day and 5 days a week will consume 2017 kWh of electricity per year, which will cost \$165.4 at an electricity cost of 8 cents/kWh.

4. **Size and Weight**
   Normally, the *smaller* and the *lighter* the heat exchanger, the better it is. This is especially the case in the *automotive* and *aerospace* industries. The space available for the heat exchanger in some cases limits the length of the tubes (heat exchanger) that can be used.

5. **Type**
   The type of heat exchanger to be selected depends primarily on the type of fluids involved; for example, a heat exchanger is suitable to cool a liquid by a gas if the surface area on the gas side is many times that on the liquid side. On the other hand, a plate or shell-and-tube heat exchanger is very suitable for cooling a liquid by another liquid.

6. **Materials**
   The materials used in the construction of the heat exchanger may be an important consideration in the selection of heat exchangers. For example, a temperature difference of 50°C or more between the tubes and the shell will pose differential thermal expansion problems and needs to be considered.